The Application of Matrix Diagonalization to the Fibonacci Sequence

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Introduction.

The Fibonacci Sequence is the infinite sequence of numbers 0,1,1,2,3,5,8,13,21,... for which the next term is found by adding the previous two terms. The sequence of Fibonacci numbers is defined by the recurrence relation

$$F_n = F_{n-1} + F_{n-2}$$

$$\text{where} \quad F_1 = F_2 = 1$$

The Fibonacci numbers occur in settings, such as biological structures of nature, music, computer science, population models and financial markets.

Fig.1

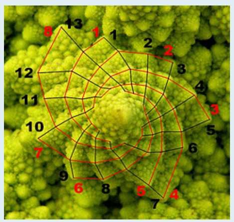

Fig.2

Fig.1 and Fig.2 are illustrations of the occurrence of Fibonacci numbers in nature.

Definition: An eigenvector of an nxn matrix A is a nonzero vector x such that Ax= λx for some scalar λ. In this case, the scalar λ is called an eigenvalue of A corresponding to the eigenvector x. The eigenvalues are the solutions to the characteristic equation det (A - λI) = 0.

Matrix Diagonalization is the process of taking a square matrix and transforming it into a diagonal matrix such that they share the same properties.

- An nxn matrix A is diagonalizable if A is similar to a diagonal matrix, that is , $A = PDP^{-1}$ for some invertible matrix P and some diagonal matrix D.

Theorem: A is an nxn matrix with n distinct eigenvalues. 1) A is diagonalizable if and only if it has n linearly independent eigenvectors. 2) $A = PDP^{-1}$, with diagonal matrix D, if and only if the columns of P are n linearly independent eigenvectors of A and the diagonal entries of D are the corresponding eigenvalues.

Method

A 2-dimensional system of linear difference equations that describe the Fibonacci Sequence:

$$\begin{bmatrix} f_2 \\ f_1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} f_1 \\ f_0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} f_{n+1} \\ f_n \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n \begin{bmatrix} 1 \\ 0 \end{bmatrix} = A^n x$$

Consider the matrix

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

Find eigenvalues: $0 = \det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - \lambda - 1 \Rightarrow \lambda = \frac{1 \pm \sqrt{5}}{2}$

Find eigenvectors:

$$\lambda_0 = \frac{1+\sqrt{5}}{2} = \left|\begin{matrix} 1 - \frac{1+\sqrt{5}}{2} & 1 & \vdots 0 \\ 1 & -\frac{1+\sqrt{5}}{2} & \vdots 0 \end{matrix}\right| = \left|\begin{matrix} 1 & -\frac{(1+\sqrt{5})}{2} & \vdots 0 \\ 0 & 0 & \vdots 0 \end{matrix}\right|$$

$$\lambda_1 = \frac{1-\sqrt{5}}{2} = \left|\begin{matrix} 1 - \frac{1-\sqrt{5}}{2} & 1 & \vdots 0 \\ 1 & -\frac{1-\sqrt{5}}{2} & \vdots 0 \end{matrix}\right| = \left|\begin{matrix} 1 & -\frac{1-\sqrt{5}}{2} & \vdots 0 \\ 0 & 0 & \vdots 0 \end{matrix}\right|$$

So our first eigenvalue-eigenvector pair is

$$\lambda_0 = \frac{1+\sqrt{5}}{2}, \quad x_0 = \begin{bmatrix} \frac{1+\sqrt{5}}{2} \\ 1 \end{bmatrix}$$

2nd eigenvector pair is

$$\lambda_1 = \frac{1-\sqrt{5}}{2}, \quad x_1 = \begin{bmatrix} \frac{1-\sqrt{5}}{2} \\ 1 \end{bmatrix}$$

Now we can construct our matrices $A = PDP^{-1}$

$$D = \begin{bmatrix} \frac{1+\sqrt{5}}{2} & 0 \\ 0 & \frac{1-\sqrt{5}}{2} \end{bmatrix} \quad P = \begin{bmatrix} \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \\ 1 & 1 \end{bmatrix}$$

using the determinant to find P inverse we find

$$P^{-1} = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 & -\frac{1-\sqrt{5}}{2} \\ -1 & \frac{1+\sqrt{5}}{2} \end{bmatrix} \quad \det(P) = \frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2} = \sqrt{5}$$

Now we plug it all in to find a formula for the nth power of A. $A^n = PD^nP^{-1}$

$$= \frac{1}{\sqrt{5}}\begin{bmatrix} \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \\ 1 & 1 \end{bmatrix}\begin{bmatrix} \left(\frac{1+\sqrt{5}}{2}\right)^n & 0 \\ 0 & \left(\frac{1-\sqrt{5}}{2}\right)^n \end{bmatrix}\begin{bmatrix} 1 & -\frac{1-\sqrt{5}}{2} \\ -1 & \frac{1+\sqrt{5}}{2} \end{bmatrix}$$

$$= \frac{1}{\sqrt{5}}\begin{bmatrix} \left(\frac{1+\sqrt{5}}{2}\right)^{n+1} & \left(\frac{1-\sqrt{5}}{2}\right)^{n+1} \\ \left(\frac{1+\sqrt{5}}{2}\right)^n & \left(\frac{1-\sqrt{5}}{2}\right)^n \end{bmatrix}\begin{bmatrix} 1 & \frac{-1+\sqrt{5}}{2} \\ -1 & \frac{1+\sqrt{5}}{2} \end{bmatrix}$$

$$= \frac{1}{\sqrt{5}}\begin{bmatrix} \left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1} & \left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n \\ \left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n & \left(\frac{1+\sqrt{5}}{2}\right)^{n-1} \left(\frac{1-\sqrt{5}}{2}\right)^{n-1} \end{bmatrix}$$

and so we can see that the formula for the nth # in the Fibonacci sequence is the lower left entry in this matrix

$$f_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right) \qquad f_{101} = 573147844013817084101$$

Results

The resulting equation is a closed-form solution for the nth term of the Fibonacci Sequence. This is an expression that allows us to calculate f_n for any n in the Fibonacci sequence.

Summary/Conclusion

Simple linear models can help us better understand systems like the Fibonacci sequence. By applying matrix diagonalization to the recursive relation we were able to build a model for the long-term direction of the Fibonacci numbers. Further analysis shows us that the ratio of the two subsequent terms in the sequence actually approaches a constant -the famous golden ratio defined as $\varphi = \frac{1+\sqrt{5}}{2}$

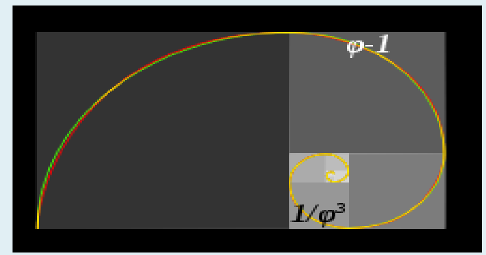

Fig.3 Logarithmic spiral whose growth factor b is related to, the golden ratio φ.

Acknowledgments

1.Lay, David C. ,2003, Linear Algebra and its Applications.

2.Notes from Dr. Jen Chang-Math247

3.Fibonacci number, 2010. *In Wikipedia, The Free Encyclopedia.* Retrieved May 6, 2010, from http://en.wikipedia.org/wiki/Fibonacci_number

4. Joseph Khoury, Application to a Probleme of Fibonacci. http://aix1.uottawa.ca/~jkhoury/fibonacci.htm