Markov Chains

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Abstract

Markov Chains are an important tool in modern applied mathematics. This tool can be use on a variety of situations such as Biology, Business, Chemistry, Engineering.

Markov Chain is described by the first-order difference equation.

$$x_{k+1} = Px_k \quad \text{for } k=0,1,2,\ldots$$

Introduction

The main purpose Markov chains (discrete random process) is to model real life situations, like calculating the probability of survival or distribution.

Stochastic Matrix, probability matrix, or transition matrix is used to describe the transitions of a Markov Chain.

There are at least three types of Stochastic Matrices

1- right stochastic matrix: Square matrix whose rows consists of nonnegative real numbers, with each row summing to 1.

2- left stochastic matrix: Square matrix whose columns consists of nonnegative real numbers whose sum is 1.

3- double stochastic matrix: Square matrix where all rows and columns sum to 1.

$$P = \begin{pmatrix} p_{1,1} & p_{1,2} & \cdots & p_{1,j} & \cdots \\ p_{2,1} & p_{2,2} & \cdots & p_{2,j} & \cdots \\ \vdots & \vdots & \ddots & \vdots & \ddots \\ p_{i,1} & p_{i,2} & \cdots & p_{i,j} & \ddots \\ \vdots & \vdots & \ddots & \vdots & \ddots \end{pmatrix} \qquad \sum_j P_{i,j} = 1$$

Scenario

Suppose a study was done on a certain county's election and determined that each year, those who voted Democratic: 70% will again vote Democratic while 20% will vote Republican and 10% will vote Libertarian. Also, for those who voted Republican: 80% will again vote Republican, while 10% will vote Democratic and 10% will vote Libertarian. Finally, for those who voted Libertarian, 40% will again vote Libertarian, while 30% will vote Republican and 30% will vote Democratic. Given the outcome of one election as 55% Democratic, 40% Republican, and 5% Libertarian. Find the likely outcome of the next two elections.

Solution

First we can define the stochastic matrix for his problem to be,

From

	D	R	L	To
P =	.70	.10	.30	D
	.20	.80	.30	R
	.10	.10	.40	L

Next we can define X_0 to be the results of the given election in 2000

$$X_0 = \begin{bmatrix} \%\text{ voting Democratic} \\ \%\text{ voting Republican} \\ \%\text{ voting Liberarian} \end{bmatrix} = \begin{bmatrix} .55 \\ .40 \\ .05 \end{bmatrix}$$

Letting X_1 represent the predicted outcome for the 2001 election, and following the form of Markov chain we obtain

$$X_1 = P X_0 = \begin{bmatrix} .70 & .10 & .30 \\ .20 & .80 & .30 \\ .10 & .10 & .40 \end{bmatrix} \begin{bmatrix} .55 \\ .40 \\ .05 \end{bmatrix} = \begin{bmatrix} .440 \\ .445 \\ .115 \end{bmatrix}$$

So the predicted outcome of the 2001 elections is as follows: 44% will vote Democrat, 44.5% will vote Republican and 11.5% will vote Libertarian

Letting X_2 represent the predicted outcome for the 2002 election, and again following the form of Markov chain we obtain

$$X_2 = P X_1 = \begin{bmatrix} .70 & .10 & .30 \\ .20 & .80 & .30 \\ .10 & .10 & .40 \end{bmatrix} \begin{bmatrix} .440 \\ .445 \\ .115 \end{bmatrix} = \begin{bmatrix} .3870 \\ .4785 \\ .1345 \end{bmatrix}$$

So the predicted outcome of the 2002 elections is as follows: 38.7% will vote Democrat, 47.85% will vote Republican and 13.45% will vote Libertarian

Conclusion

Here the Markov chain is calculated through simple matrix multiplication yet it has a powerful effect of predicting the outcome of future elections. Here we only took the equation two steps forward, but it is also interesting to study the long term behavior of Markov chain.

Acknowledgements

Lay, David C. *Linear Algebra and Its Applications*. Third ed. Pearson Education, 2007. Print.

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