Arnold's Cat Map

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Introduction:

The purpose of this poster is to introduce the mapping know as Arnold's Cat Map, discovered by Russian mathematician Vladimir Arnold, and to explain some its properties using linear algebra. This mapping is a simple illustration of some of the principles of chaos theory – specifically, showing the underlying order to an apparently random evolution of a system. In this example, an image is hit with a transformation that apparently randomizes the original organization of its pixels. However, if iterated enough times, the original image reappears.

Methods:

An image is composed of discrete units called pixels. A pixel is a small square representing some color value, which when taken together form the mosaic that is the image. Then Arnold's Cat Map transformation is defined as:

$$\Gamma : \mathbb{R}^2 \mapsto \mathbb{R}^2 \text{ where}$$

$$\Gamma \begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} x+y \\ x+2y \end{bmatrix} \bmod n$$

To better understand the mechanism of the transformation Γ, we can decompose it into its elemental pieces and see what it does to an image of a cat.

1. Shear in the x-direction by a factor of 1.

$$\begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} x+y \\ y \end{bmatrix}$$

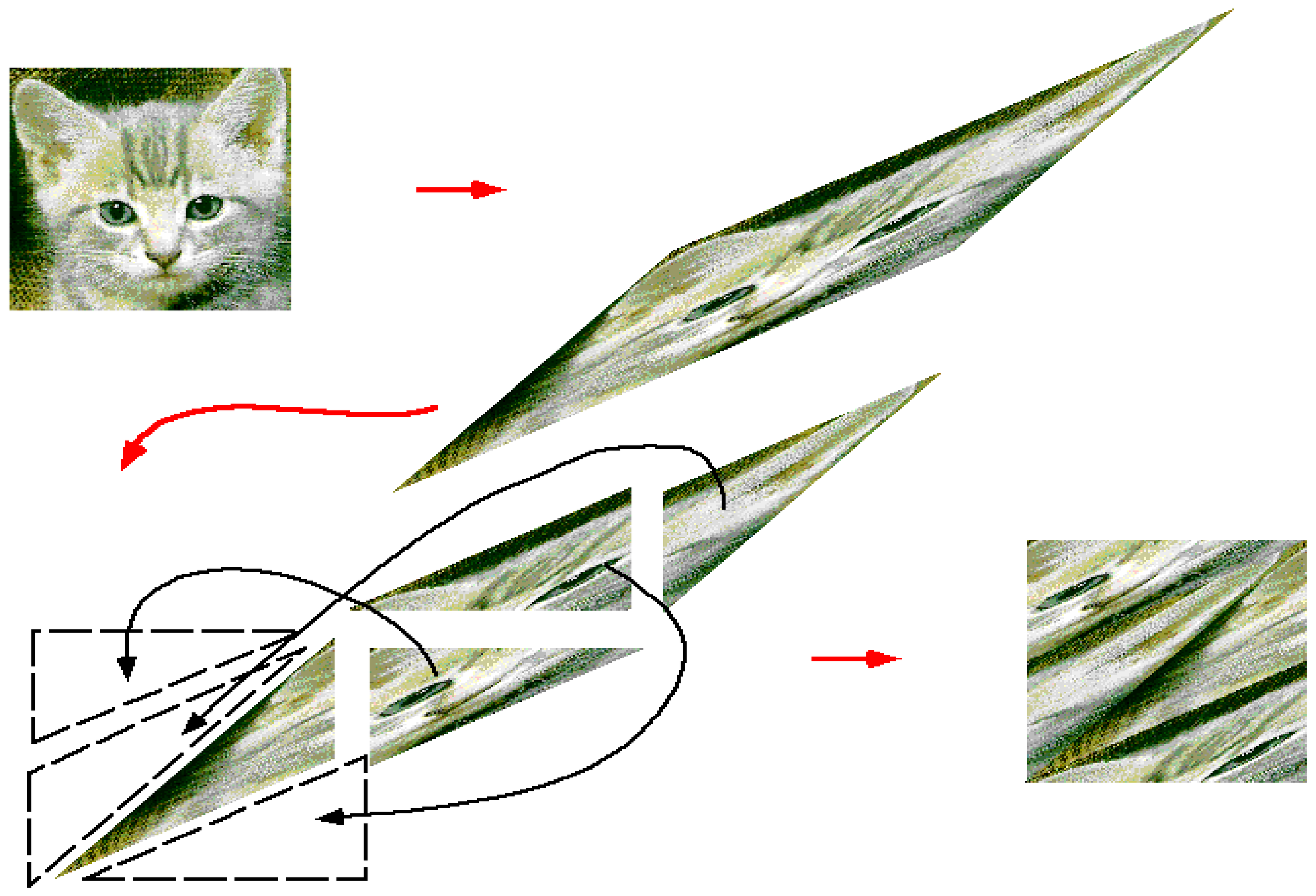
2. Shear in the y-direction by a factor of 1.

$$\begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} x \\ x+y \end{bmatrix}$$

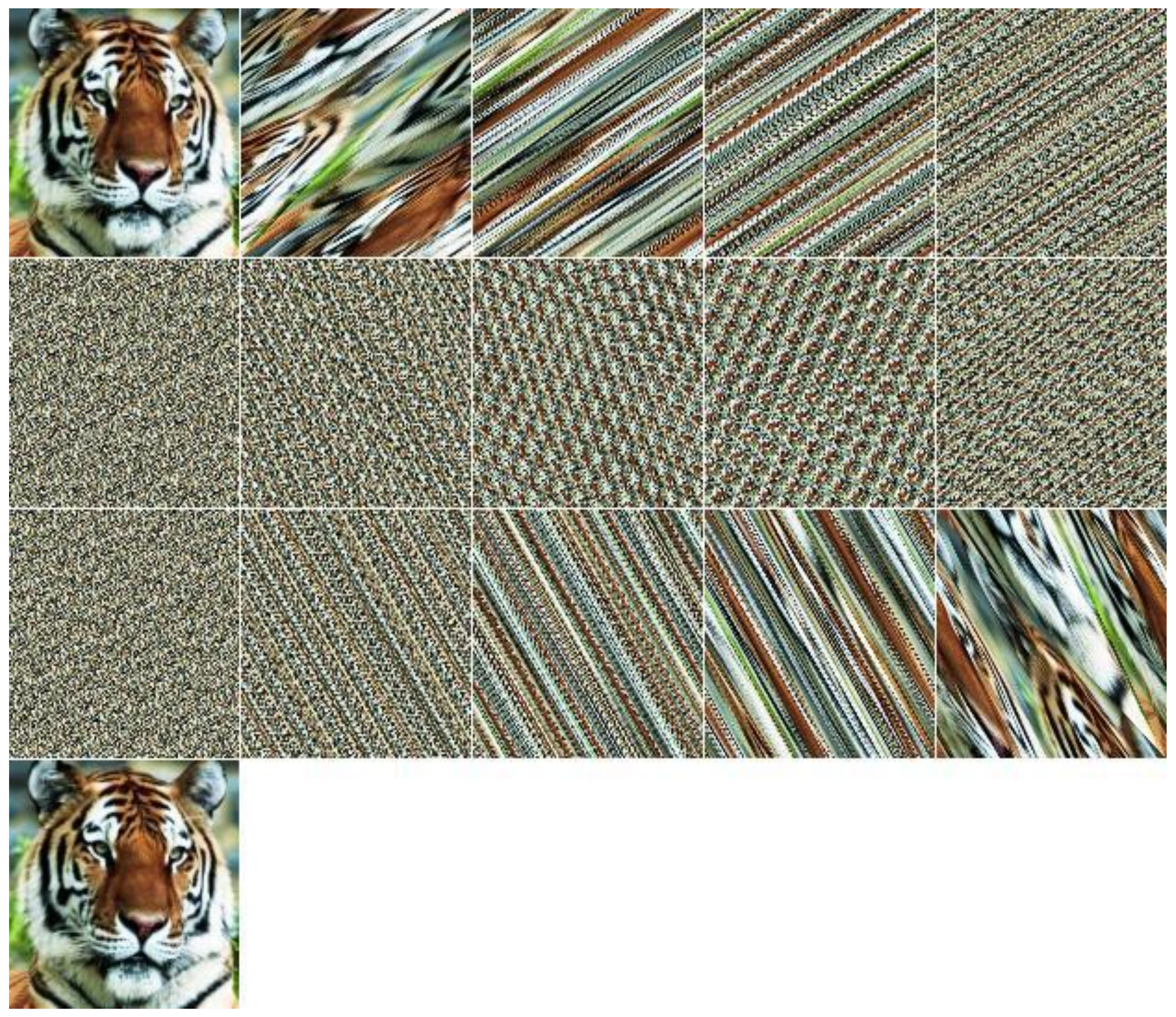
3. Evaluate the modulo.

$$\begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} x \\ y \end{bmatrix} \bmod n$$

The first step shows the shearing in the x and y directions, followed by the evaluation of the modulo operation, and finally the reassembly of the image in its new form:



The sequence above is one iteration of Arnold's Cat Map. Below is an example where the mapping is applied repeatedly onto a 124 by 124 pixel image of a tiger, which results in a surprising thing.



Initially the image dissolves into a television-static like state, and then eventually on the fifteenth iteration it reforms back into the original image.

Results:

Why does order emerge out of this apparently chaotic mapping? The simplest approach would be to examine a single pixel during the mapping process shown below:

$$\Gamma \begin{bmatrix} 32 \\ 13 \end{bmatrix} = \begin{bmatrix} 32 + 13 \\ 13 + 2 \bullet 32 \end{bmatrix} \bmod 124 = \begin{bmatrix} 45 \\ 58 \end{bmatrix} \to \begin{bmatrix} 103 \\ 37 \end{bmatrix} \to \begin{bmatrix} 16 \\ 53 \end{bmatrix} \to \begin{bmatrix} 69 \\ 122 \end{bmatrix}$$

$$\to \begin{bmatrix} 67 \\ 65 \end{bmatrix} \to \begin{bmatrix} 8 \\ 73 \end{bmatrix} \to \begin{bmatrix} 81 \\ 30 \end{bmatrix} \to \begin{bmatrix} 111 \\ 17 \end{bmatrix} \to \begin{bmatrix} 4 \\ 21 \end{bmatrix} \to \begin{bmatrix} 25 \\ 46 \end{bmatrix} \to \begin{bmatrix} 71 \\ 117 \end{bmatrix}$$

$$\to \begin{bmatrix} 64 \\ 57 \end{bmatrix} \to \begin{bmatrix} 121 \\ 54 \end{bmatrix} \to \begin{bmatrix} 51 \\ 105 \end{bmatrix} \to \begin{bmatrix} 32 \\ 13 \end{bmatrix}$$

As you can see, after fifteen iterations, the pixel – as would any other pixel in the image - has returned to its initial position. This agrees with the earlier observation with the complete 124 by 124 pixel image of the tiger.

Conclusions:

Even though there is no practical use for the mapping known as Arnold's Cat Map, it is an interesting illustration of how something which appears completely random and without any sense of order at all can somehow morph itself back into a state of order seemingly out of the blue. Which seemingly goes against the basic laws of nature which state that entropy and disorder only increase over time.

Acknowledgements:

Arnold's Cat Map, Gabriel Peterson, College of the Redwoods, 1997