How to be an attentive parent:

Using the Simplex Algorithm to find an optimal solution to a Linear Programming problem.

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Introduction

Imagine that you're a parent with three teenage kids at home. It's hard enough to get them all in the same room together, let alone doing the same thing at the same time. You're very busy, as parents tend to be, but you want to make sure you spend enough quality time with each of your kids every week. You also need to show each of them an equal amount of time and attention. You've narrowed the family activity options down to a small handful, but none of them hold everyone's attention for long. Given those limitations, **what is the *most efficient* way of spending time on family activities?**

This dilemma can be easily modeled as a Linear Programming problem, and an optimal solution can be found using the Simplex Algorithm.

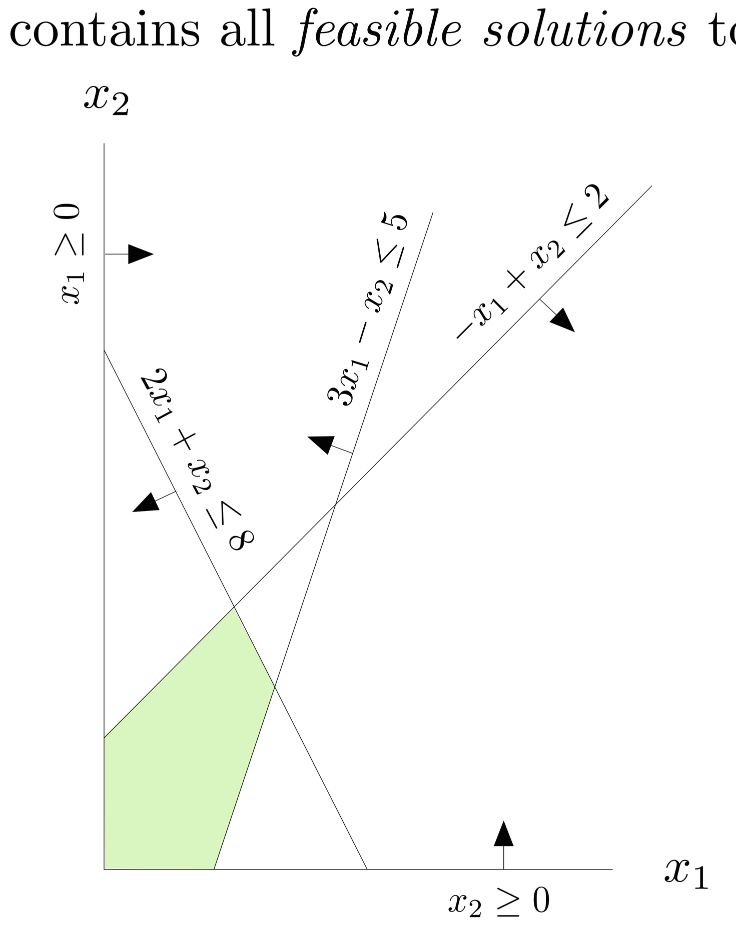
Methods

A ***Linear Programming*** problem is one in which your goal is to *maximize* or *minimize* a linear equation of n variables (called the ***objective function***), subject to a set of m limits on those variables in the form of *linear inequalities*. As a rule, no variable can take on a negative value.

To reinforce the idea, here is a simple example in 2 variables and 3 limits ($n = 2, m = 3$):

$$\begin{aligned} \textbf{maximize:} \quad & x_1 + x_2 \\ \textbf{subject to:} \quad & 2x_1 + x_2 \leq 8 \\ & 3x_1 - x_2 \leq 5 \\ & -x_1 + x_2 \leq 2 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Graphically, you can imagine each of these linear inequalities as *half-spaces* that divide a 2-dimensional Cartesian graph into "valid" and "invalid" regions. The intersection of all of the valid half-spaces is called the ***simplex*** or ***feasible region***, and it is this region that contains all *feasible solutions* to the linear program.



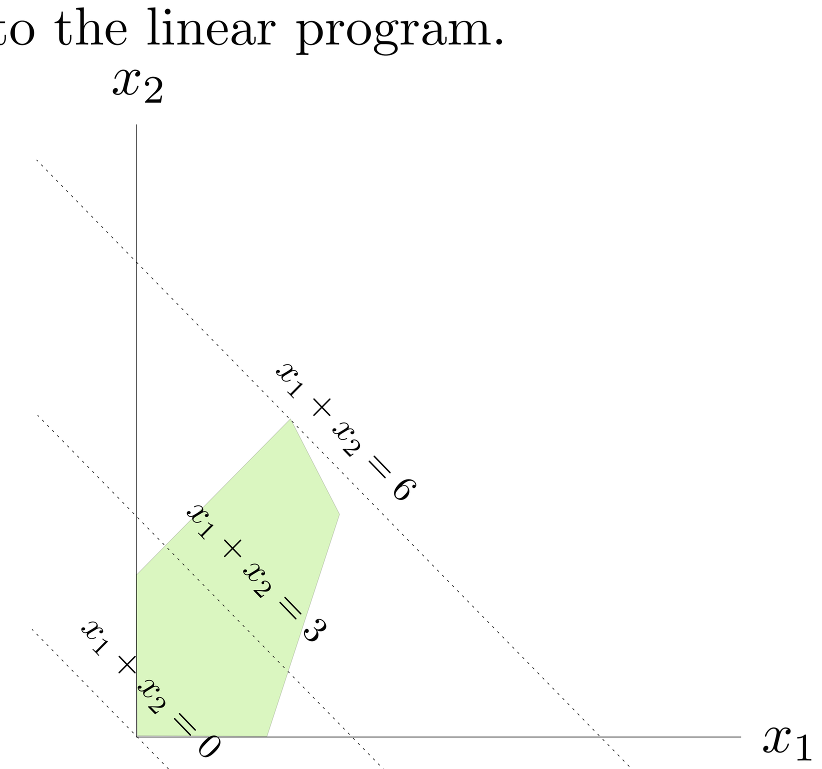


Figure 1 *(left)* shows the simplex for this example.
Figure 2 *(above)* shows different potential values for the objective function, which we are trying to maximize.

The ***Simplex Algorithm*** relies on two key insights about the geometry of a feasible region *(for which proofs have been found)*:

- the feasible region is convex, so any local maximum value is also the global maximum value of the objective function.
- an optimal solution will occur on one or more vertices of the feasible region.

The simplex algorithm begins by finding any vertex of the feasible region. It then "crawls along" the edges of the feasible region to adjacent vertices such that the value of the objective function at the adjacent vertex is *not smaller than* the current value. This continues until the algorithm cannot possibly crawl to a better value of the objective function, at which point an optimal solution to the problem has been found.

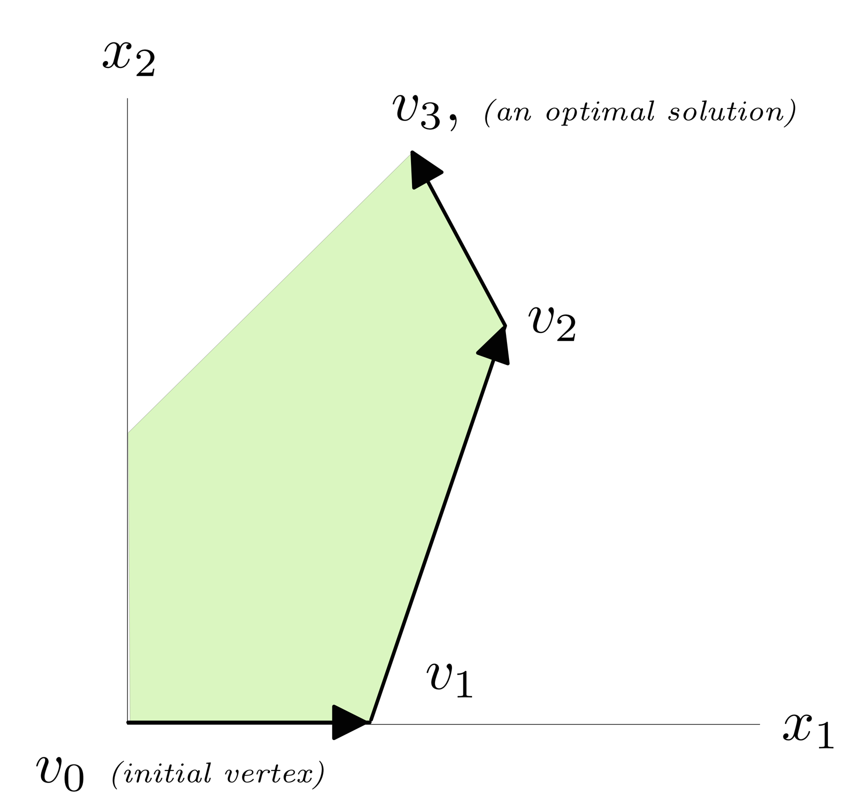


Figure 3 shows a possible path of the Simplex Algorithm from an initial vertex to an optimal solution. Notice how the algorithm traverses the edge of the feasible region, and that from v_3 along all possible edges to any adjacent vertex, the value of the objective function decreases.

To accomplish this "crawling" action, the algorithm chooses a variable with positive coefficient from the objective function, and increases the value of that variable as much as possible without invalidating any of the limits. One of the limits will have tighter restrictions than the others, and that limit is then solved for the chosen variable. The entire linear program is then *pivoted* on that variable, replacing all occurrences with its new value. This will negate coefficients in the objective function, so after a few repetitions of this procedure, we're left with a linear program that cannot possibly be optimized further from its current state. An optimal solution is then obvious.

Results

To solve our problem of finding the ideal set of family activites, let's first define our goals, our activity options, and the kids' attention spans for each activity.

Goals:

- **Everyone gets at least 4 quality hours every week.**
- **Nobody gets over a half hour more or less than anyone else.**

Attention Span per Hour of Activity:

	cooking	gaming	swimming
Liz	30 min	1 hour	0
Frank	48 min	0	5 min
Gwen	0	30 min	55 min

From this information, we can already write our problem as a Linear Programming problem:

$$\begin{aligned} \textbf{minimize:} \quad & c + g + s \\ \textbf{subject to:} \quad & 30c + 60g \geq 240 \\ & 48c + 5s \geq 240 \\ & 30g + 55s \geq 240 \\ & (30c + 60g) - (48c + 5s) \leq 30 \\ & (30c + 60g) - (48c + 5s) \geq -30 \\ & (30c + 60g) - (30g + 55s) \leq 30 \\ & (30c + 60g) - (30g + 55s) \geq -30 \\ & (48c + 5s) - (30g + 55s) \leq 30 \\ & (48c + 5s) - (30g + 55s) \geq -30 \\ & c, g, s \geq 0 \end{aligned}$$

Running the simplex algorithm on our problem, we find that an optimal solution is given by **4 hours, 38 minutes** of cooking, **1 hour, 40 minutes** of gaming, and **3 hours, 27 minutes** of swimming every week, for a total of 9 hours, 45 minutes. We can also verify that everyone gets almost exactly **4 hours** of time per week, well within our "equality" limits.

Conclusions

The Simplex Algorithm is an amazingly powerful tool for solving a wide range of problems. This tongue-in-cheek example hints at the applicability the algorithm may have to real-world problems. The algorithm is simple and fast enough to implement for most any device, and so it could easily be a ubiquitious tool in your arsenal. You need only recognize and be able to setup a linear programming problem when in the presence of one.

Acknowledgements & References

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