Chaos

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What is **Chaos?**

When most of us think of chaos we may think of Rion trying to track stand on his bike, riots or perhaps California weather; basically an utter random/disorganized mess. Taken from Dictionary.com :

a state of utter confusion or disorder; a total lack of organization or order.

This, although related in many ways, is not the chaos I am referring to. The science and math of chaos is the overlying theme of this poster. So then, what exactly is the “science and math” of chaos?

The Science and Math of **Chaos**

The science and math of chaos saw its first light in an experiment carried out by Edward Lorenz, a meteorologist at MIT in 1960. With the help from a computer he created a simulation of the weather given certain initial conditions. When he re-entered the initial conditions with a degree of six decimal places instead of three he discovered remarkable changes…something totally different from the first results…chaos.

Unlike the chaos we all know mathematical chaos is somewhat deterministic; this does go against the definition of chaos, but in chaotic systems there is a sort of unpredictability not shown in system that are deterministic.

The word chaos was first used in a mathematical sense in “Period Three Implies Chaos”, a paper written by American mathematicians James Yorke and Tien-Yien Li in 1975. But the simplest way to understand this “scientific” chaos was developed by a famous Russian mathematician named Vladimir I. Arnold, in an experiment called “Arnold’s cat map”. In Arnold’s cat map we deal with a matrix (that represents a picture) that undergoes a transformation that basically distorts the original picture to a mix of black and white randomness, but eventually bringing it back to the original picture.

$$\Gamma\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \bmod 1$$

This is the matrix that, when applied to a pixilated version of a picture will transform it into a mess and eventually bring it back to the original image.

Apply the who to that what?

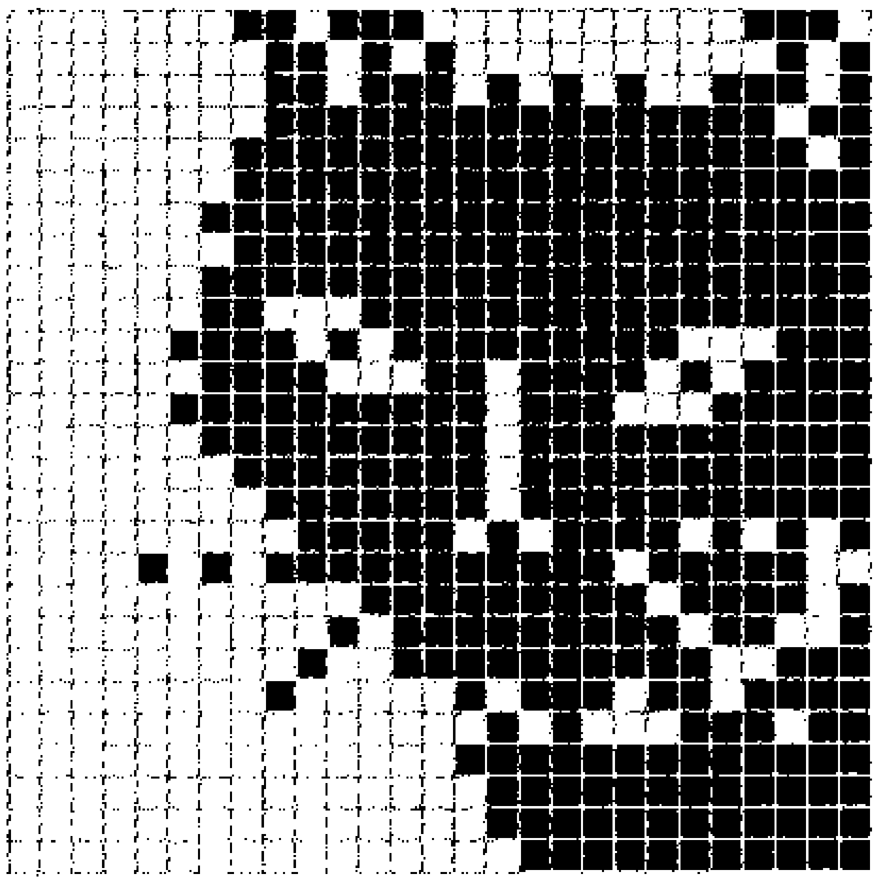
You may be confused when I say “apply the two matrices to the picture”…how can you apply a matrix to a picture?

A picture is worth a thousand words

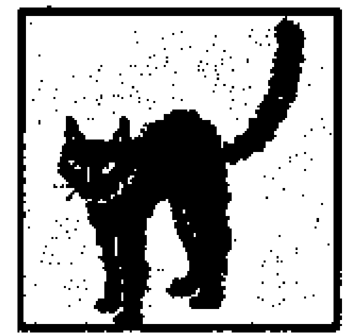
You may be familiar with the saying made famous by Fred R. Barnard, but it should be read:

A picture is worth a thousand pixels

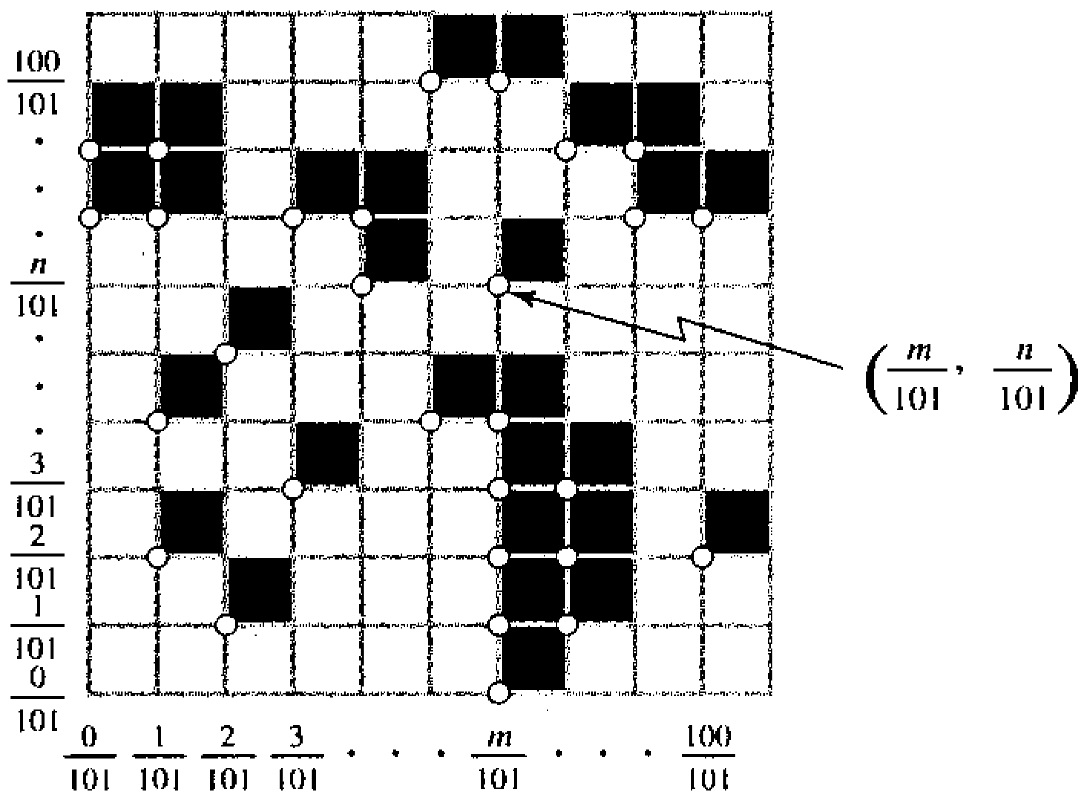
Pixel? What is a pixel? Well think of a picture as thousands and thousands of tiny colored squares. Here is an example:



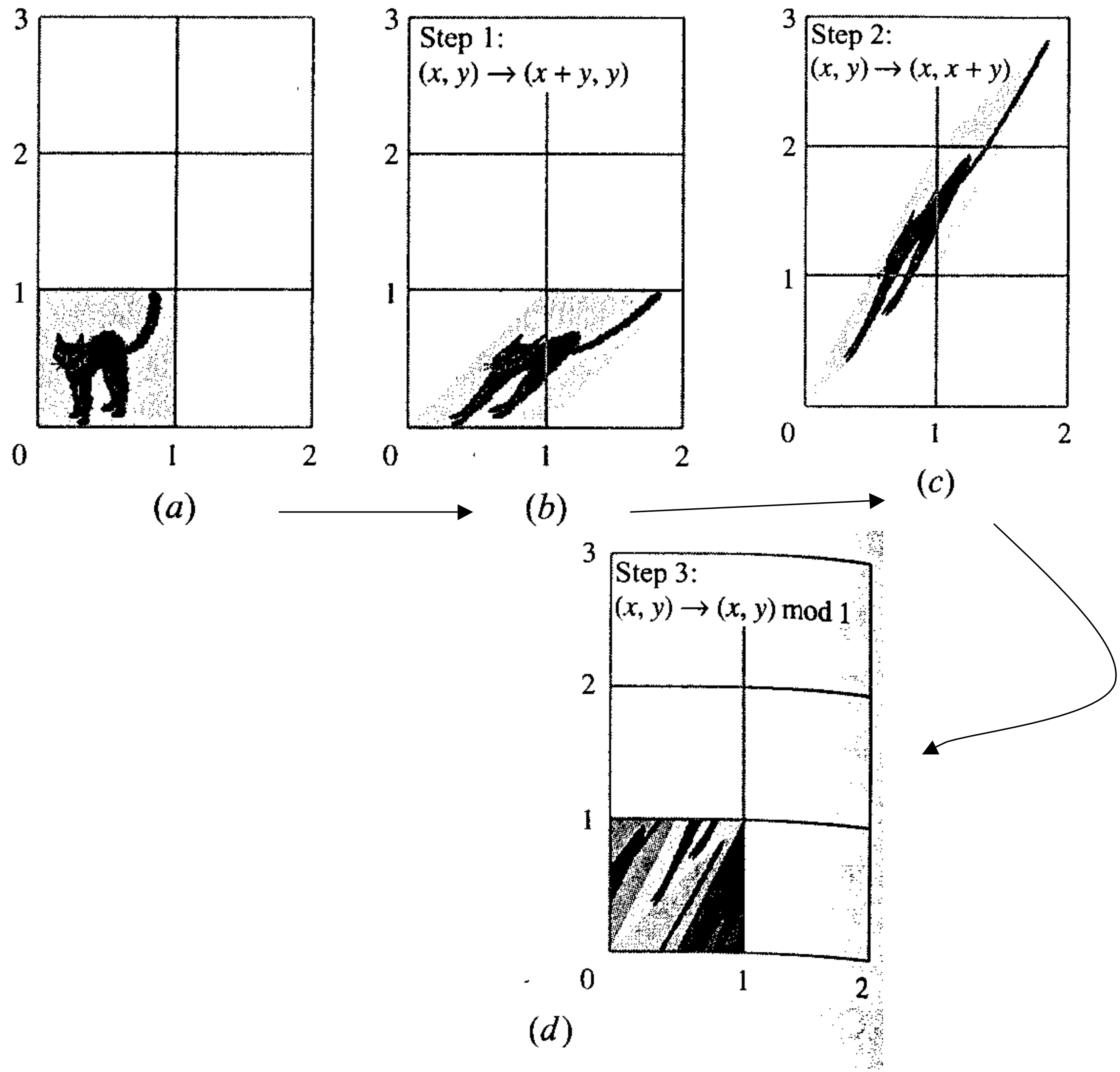
Enlarged view of cat's face showing individual pixels



When the thousands of pixels are combined to create a picture this is known as a pixel map and looking at the pixilated version of the cat's head one might be able to see how all this chaos stuff relates to Linear Algebra…The pixel map looks like a matrix! Every tiny square (pixel) of the pixel map represents a number, so really a picture (on your computer or TV screen) is really just a matrix.



This matrix you see is a 101x101. The variables *m* and *n* represent integers ranging from 0 to 100.



Step 1:
$(x, y) \to (x + y, y)$
Step 2:
$(x, y) \to (x, x + y)$
Step 3:
$(x, y) \to (x, y) \bmod 1$
(a)
(b)
(c)
(d)

Step 1. Shear in the x-direction with factor 1 (Figure 11.15.1b):

$$(x, y) \to (x + y, y)$$

or in matrix notation

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + y \\ y \end{bmatrix}$$

Step 2. Shear in the y-direction with factor 1 (Figure 11.15.1c):

$$(x, y) \to (x, x + y)$$

or, in matrix notation,

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ x + y \end{bmatrix}$$

Step 3. Reassembly into S (Figure 11.15.1d):

$$(x, y) \to (x, y) \bmod 1$$

Modulus arithmetic

To fully understand how this whole chaos things works we need to understand modulus arithmetic. What modulus arithmetic basically does is it keeps the image we are doing work on in the same space it already occupies and does not allow it to go beyond its bounds. The modulus we are going to be using in chaotic systems is mod 1. Here is an example:

$$2.3 \bmod 1 = 0.3, \quad 0.9 \bmod 1 = 0.9, \quad -3.7 \bmod 1 = 0.3, \quad 2.0 \bmod 1 = 0$$

x mod y

So… when you divide x by y the remainder is the new number. As you may or may not be able to see, this number (when the modulus is 1) cannot be larger then 1 nor can it be smaller then 0. So any number you take the modulus of 1 of is going to be on an interval [0,1). So if we are using an xy axis as our plane our picture is sitting on it will all be concentrated in a unit square.

Making any sense? This *is* **chaotic** after all…

To make things a little easier we will look at just two points with a small period…

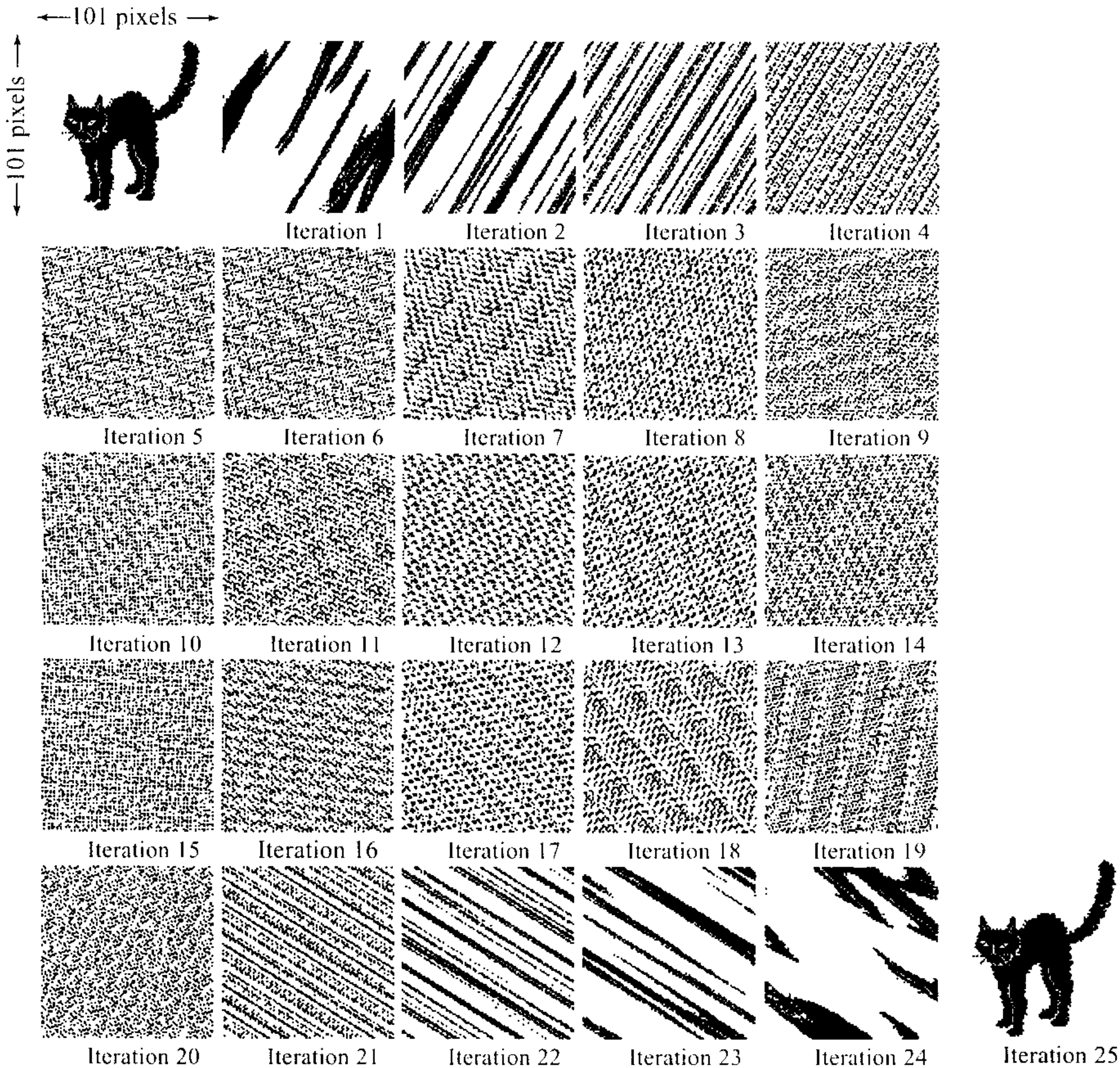
If $p = 76$, then (2) becomes

$$\Gamma\left(\begin{bmatrix} \dfrac{m}{76} \\[2mm] \dfrac{n}{76} \end{bmatrix}\right) = \begin{bmatrix} \dfrac{m+n}{76} \\[2mm] \dfrac{m+2n}{76} \end{bmatrix} \bmod 1$$

In this case the successive iterates of the point $\left(\frac{27}{76}, \frac{58}{76}\right)$ are

$$\begin{array}{ccccccccccccccccc}
0 & & 1 & & 2 & & 3 & & 4 & & 5 & & 6 & & 7 & & 8 \\
\begin{bmatrix} \frac{27}{76} \\ \frac{58}{76} \end{bmatrix} & \rightarrow & \begin{bmatrix} \frac{9}{76} \\ \frac{67}{76} \end{bmatrix} & \rightarrow & \begin{bmatrix} \frac{0}{76} \\ \frac{67}{76} \end{bmatrix} & \rightarrow & \begin{bmatrix} \frac{67}{76} \\ \frac{58}{76} \end{bmatrix} & \rightarrow & \begin{bmatrix} \frac{49}{76} \\ \frac{31}{76} \end{bmatrix} & \rightarrow & \begin{bmatrix} \frac{4}{76} \\ \frac{35}{76} \end{bmatrix} & \rightarrow & \begin{bmatrix} \frac{39}{76} \\ \frac{74}{76} \end{bmatrix} & \rightarrow & \begin{bmatrix} \frac{37}{76} \\ \frac{35}{76} \end{bmatrix} & \rightarrow & \begin{bmatrix} \frac{72}{76} \\ \frac{31}{76} \end{bmatrix}
\end{array}$$

Recall that the number in the denominator represents how big the matrix is…so this matrix is a 76x76 and if there were a pixel at the point (27/76,58/76) it would return to it's original position after 9 iterations, or, to put it in simpler terms it would have a period of 9.


←101 pixels →
←101 pixels →
Iteration 1
Iteration 2
Iteration 3
Iteration 4
Iteration 5
Iteration 6
Iteration 7
Iteration 8
Iteration 9
Iteration 10
Iteration 11
Iteration 12
Iteration 13
Iteration 14
Iteration 15
Iteration 16
Iteration 17
Iteration 18
Iteration 19
Iteration 20
Iteration 21
Iteration 22
Iteration 23
Iteration 24
Iteration 25


Chuck Norris
FIGHT CLUB
Only Tap Water Delivers
SM
The End
Weather: cool
Food: 1220 pounds
Next landmark: 39 miles
Miles traveled: 265 miles
GAMESPOT