Constructing Curves and Surfaces through Specified Points

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Introduction:

Using linear algebra, it is possible to find the equation of a curve or line if given specific data points that lie on the curve. The amount of points needed depends on how complicated the curve is. This is done by creating a linear system whose variables are the coefficients of the general equation being solved for. Because a linear system has a non-trivial solution if and only if the determinant is equal to zero, we are able to solve it and get an equation that passes through the points.

How to:

1) Create a linear system including the general equation of the curve or surface containing general points (x,y) and the points given.

Type:	Equation:
Line	$ax+by+c=0$
Circle	$ax^2+by^2+cx+dy+e=0$
Conic Section	$ax^2+by^2+cxy+dx+ey+f=0$

2) Set the determinant of the corresponding matrix equal to zero.

3) Calculate the determinant with which ever method is most convenient. Solve for x and y.

Results:

You are left with an equation of a curve or surface passing through the original specified points. You can check your work by plugging in the points and confirming the equation is true.

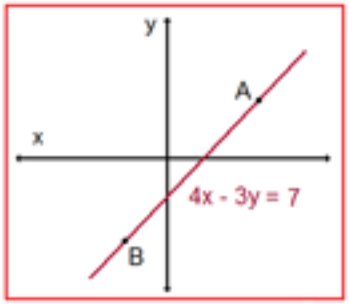
Example 1

Given the points A = (4,3) and B = (-2,-5) find the equation of a line passing through both points.

By entering the points A and B, and a general point (x,y), we can make the following linear system:

$$\begin{aligned} ax + by + c &= 0 \\ 4a + 3b + c &= 0 \\ -2a + -5b + c &= 0 \end{aligned}$$

$$A = \begin{bmatrix} x & y & 1 \\ 4 & 3 & 1 \\ -2 & -5 & 1 \end{bmatrix} \quad X = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$



The equation AX = B has a non-trivial solution if and only if det(A) = 0.

$$\det(A) = [(x\cdot3\cdot1 + y\cdot1\cdot-2 + 1\cdot4\cdot-5) - (1\cdot3\cdot-2 + y\cdot4\cdot1 + x\cdot1\cdot-5)]$$

$$\det(A) = 8x - 6y - 14 = 4x - 3y - 7 = 0$$

Example 2

Given the points A=(0,0) B=(0,-1) C=(2,0) D=(2,-5) and E=(4,-1) find the equation of a conic section passing through these points.

Create a linear system:

$$\begin{aligned} ax^2 + by^2 + cxy + dx + ey + f &= 0 \\ 0a + 0b + 0c + 0d + 0e + f &= 0 \\ 0a + 1b + 0c + 0d - 1e + f &= 0 \\ 4a + 0b + 0c + 2d + 0e + f &= 0 \\ 4a + 25b - 10c + 2d - 5e + f &= 0 \\ 16a + 1b - 4c + 4d - 1e + f &= 0 \end{aligned}$$

So:

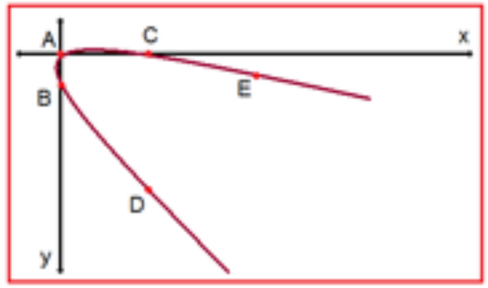
$$A = \begin{bmatrix} x^2 & y^2 & xy & x & y & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -1 & 1 \\ 4 & 0 & 0 & 2 & 0 & 1 \\ 4 & 25 & -10 & 2 & -5 & 1 \\ 16 & 1 & -4 & 4 & -1 & 1 \end{bmatrix} \quad X = \begin{bmatrix} a \\ b \\ c \\ d \\ e \\ f \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Use co-factor expansion to get:

$$\det(A) = x^2 \begin{vmatrix} 0 & 2 & 0 \\ -10 & 0 & -5 \\ -4 & -4 & -1 \end{vmatrix} - xy \begin{vmatrix} 4 & 2 & 0 \\ 0 & 0 & -5 \\ 0 & -4 & -1 \end{vmatrix} + x \begin{vmatrix} 4 & 0 & 0 \\ 0 & -10 & -5 \\ 0 & -4 & -1 \end{vmatrix} - y \begin{vmatrix} 4 & 0 & 2 \\ 0 & -10 & 0 \\ 0 & -4 & -4 \end{vmatrix}$$

$$+ x^2 \begin{vmatrix} 0 & 0 & 2 \\ 25 & -10 & 0 \\ 1 & -4 & -4 \end{vmatrix} - y^2 \begin{vmatrix} 4 & 0 & 2 \\ 0 & -10 & 0 \\ 0 & -4 & -4 \end{vmatrix} + xy \begin{vmatrix} 4 & 0 & 2 \\ 0 & 25 & 0 \\ 0 & 1 & -4 \end{vmatrix} - x \begin{vmatrix} 4 & 0 & 0 \\ 0 & 25 & -10 \\ 0 & 1 & -4 \end{vmatrix}$$

$$\det(A) = -160x^2 - 160y^2 - 320xy + 320x - 160y = -x^2 - y^2 - 2xy + 2x - y = 0$$



Summary:

We are able to calculate the equation of a curve or surface given points from the fundamental linear algebra theorem that states a matrix will have a non-trivial (non-zero) solution if its determinant is equal to zero. A matrix is created by forming a system of equations that consists of general equations depending on the type of curve with the given points plugged into it. Simplifying will result in an equation describing the curve including the points specified.

Conclusion:

This application of linear algebra provides a generally simple way to find equations relating specific points on a graph. For simple applications such as finding the equation of a line connecting two points this method may take more work. However, when considering more complicated curves such as conic sections this application can simplify the calculations involved by requiring you to simply calculate the determinant and solve for the variables x and y.

Sources:

- abel.math.harvard.edu/archive/21b_summer_05/supplements/curves_surfaces.pdf

- Lay, David C. Linear Algebra and Its Applications (3rd ed.), Addison Wesley, ISBN 978-0321287137

- http://www.math.ucdavis.edu/~daddel/linear_algebra_appl/Applications/curve_schetch/curve_schetch/