# Modeling Physical Systems: Coupled Differential Equations

Jesse Burgess

## Introduction

In physics, we often attempt to model a physical system mathematically. Generally we wish to be able to predict the state of the system at some time in the future, given the present state of the system. For a single object, we can accomplish this by writing a differential equation that describes the motion of the object and then solving this equation. However, when the system contains multiple *interacting* components, we must write a differential equation for each component and then solve all equations *simultaneously*. Fortunately, by utilizing the techniques of linear algebra, we can instead rewrite our equations such that they may be solved individually, one at a time.

## Methods

We will demonstrate this method by considering a one-dimensional system with two interacting objects:

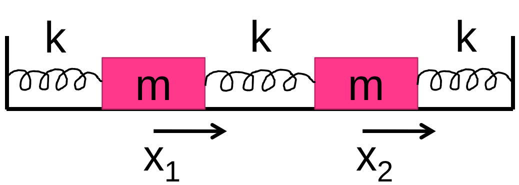


**Figure 1: Two identical masses attached to each other and to the boundaries by identical springs.**

### Equations of Motion

By using Newton's second law together with Hooke's law, we get the following equations of motion for the positions of the two masses:

$$\ddot{x}_1 = -\frac{2k}{m}x_1 + \frac{k}{m}x_2, \quad \ddot{x}_2 = \frac{k}{m}x_1 - \frac{2k}{m}x_2$$

where the double-dot notation indicates second-order differentiation with respect to time. The difficulty with these equations is that they are *coupled*: both variables appear in both equations. Thus the equations must be solved simultaneously.

### Write in Matrix Form

$$\begin{bmatrix}\ddot{x}_1\\ \ddot{x}_2\end{bmatrix} = \begin{bmatrix}-2k/m & k/m\\ k/m & -2k/m\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

In our current basis, the two components of the position vector represent the individual positions of the masses. This basis is a natural choice from a physical point of view, but is not mathematically the most convenient. We will instead use the eigenvectors of the above matrix as our basis, allowing us to diagonalize the matrix and thus separate the differential equations.

### Diagonalize

Solving the eigenvalue problem for our matrix, we get the following eigenvalues and normalized eigenvectors:

$$\lambda_I = -\frac{k}{m} \qquad \lambda_{II} = -\frac{3k}{m}$$

$$v_I = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\ 1\end{bmatrix} \qquad v_{II} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\ -1\end{bmatrix}$$

In this basis our matrix equation becomes:

$$\begin{bmatrix}\ddot{x}_I\\ \ddot{x}_{II}\end{bmatrix} = \begin{bmatrix}-k/m & 0\\ 0 & -3k/m\end{bmatrix}\begin{bmatrix}x_I\\ x_{II}\end{bmatrix}$$

giving us the following equations of motion:

$$\ddot{x}_I + \frac{k}{m}x_I = 0 \qquad \ddot{x}_{II} + \frac{3k}{m}x_{II} = 0$$

Note that the variables are no longer 'mixed:' each equation contains only one variable, allowing us to use standard methods for solving linear differential equations.

### Solve

We take the initial velocities to be zero, leaving only initial positions. With these conditions, the solutions to the above equations are:

$$x_I(t) = x_I(0)\cos\sqrt{\frac{k}{m}}t \qquad x_{II}(t) = x_{II}(0)\cos\sqrt{\frac{3k}{m}}t$$

But remember, these solutions are in a basis where $x_I$ and $x_{II}$ do not represent the positions of the individual masses. Instead, $x_I$ and $x_{II}$ are in the 'direction' of the eigenvectors $v_I$ and $v_{II}$. To get solutions which correspond to the actual positions of the masses, we project the vector containing these solutions back onto our original basis.

## Results

Upon projection, we finally obtain the solutions to our original equations of motion:

$$x_1(t) = \tfrac{1}{2}[x_1(0) + x_2(0)]\cos\sqrt{\tfrac{k}{m}}t + \tfrac{1}{2}[x_1(0) - x_2(0)]\cos\sqrt{\tfrac{3k}{m}}t$$

$$x_2(t) = \tfrac{1}{2}[x_1(0) + x_2(0)]\cos\sqrt{\tfrac{k}{m}}t - \tfrac{1}{2}[x_1(0) - x_2(0)]\cos\sqrt{\tfrac{3k}{m}}t$$

## Summary

By representing a set of coupled differential equations in matrix form and by projecting them onto their eigenbasis, we were able to separate the equations and solve them individually.

## Conclusions

As we have seen, the methods of linear algebra are readily adaptable to physical applications, largely due to the versatility of the concept of a 'vector,' which needn't necessarily represent a position in real space. Such methods are widely used in mechanics.

**Acknowledgments:** This sample problem was developed with the aid of *Principles of Quantum Mechanics* by R. Shankar (Springer Science+Business Media, LLC, New York, 1994)