Games of Strategy

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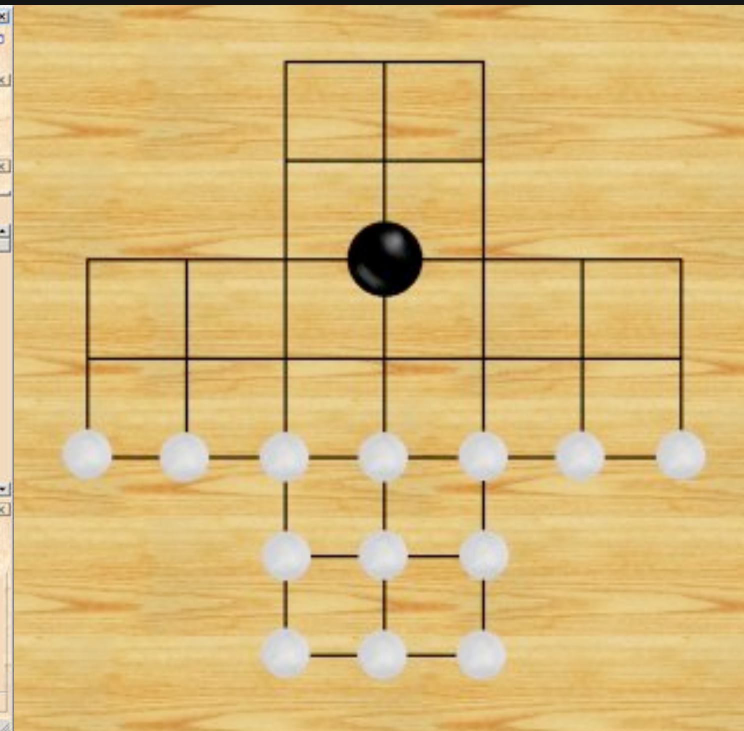
Introduction

Games of Strategy are games in which the individual player's decisions have great importance in determining the outcome of the game.

Examples of strategy games you may have heard of

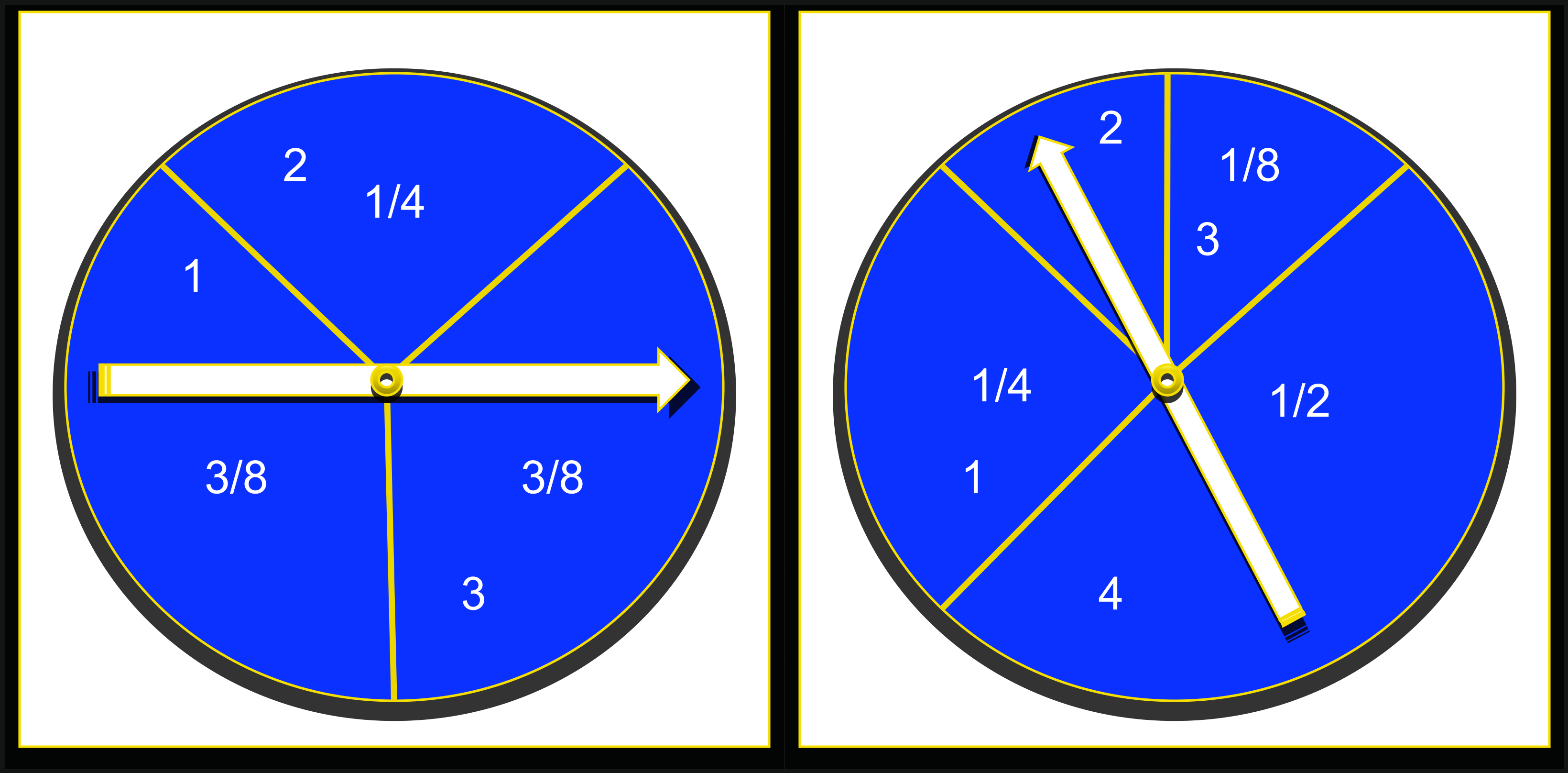
Board Games:
- Checkers
- Chess
- Tick Tack Toe

Computer/Video Games:
- Command & Conquer (RTS)



The use of Linear Algebra is a very useful tool in games of strategy.
With the use of matrix techniques we will find a way to find the best strategy for the players in a general game where the players compete for particular objectives.

Consider the following carnival-type game to demonstrate the basic concepts of game theory.



- Each player has a stationary wheel with a movable pointer on it
- Player 1 has a wheel that is divided into three sectors, while Player 2 has a wheel divided into 4 sectors.
- Each player spins the pointer of their respective wheel
- Depending on the move each player makes, Player 2 then makes some form of payment to Player 1 according to the table below

Player 2

Player 1

	1	2	3	4
1	\$2	\$0	-\$1	-\$2
2	-\$3	\$6	-\$3	-\$4
3	\$4	-\$5	\$5	\$8

- Notice some of the entries are negative, indicating that Player 1 makes a payment to Player 2 o The game described above is an example of a **two-person zero-sum matrix game**

o In a general game of this type, let Player 1 have ***m*** possible moves and let Player 2 have ***n*** possible moves. In a game after each player makes a move a **payoff** is made from Player 2 to Player 1

For $i = 1, 2, \ldots, m$, and $j = 1, 2, \ldots, n$, let us set
A_{ij} = payoff that Player 2 makes to Player 1 if Player 1 makes move i and Player C makes move j

We arrange these mn possible payments in the form of an m x n matrix, which we will call the ***payoff matrix*** of the game

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

Each Player makes their move on a probabilistic basis, and in the general case we contribute the following definitions

p_i = probability that Player 1 makes move i (i = 1, 2,..., m)

q_j = probability that Player 2 makes move j (j = 1, 2,..., n)

It follows that

$$p_1 + p_2 + \ldots + p_m = 1$$

and

$$q_1 + q_2 + \ldots + q_n = 1$$

Now with the probabilities p_i and q_j we form two vectors

$$\mathrm{p} = \begin{bmatrix} p_1 & p_2 & . & . & . & p_m \end{bmatrix} \quad \text{and} \quad \mathrm{q} = \begin{bmatrix} q_1 \\ q_2 \\ . \\ . \\ . \\ q_n \end{bmatrix}$$

Now if we multiply each possible payoff by its corresponding probability & sum over all possible payoffs then we get

$$a_{11}p_1q_1 + a_{12}p_1q_2 + \ldots + a_{1n}p_1q_n + a_{21}p_2q_1 + \ldots + a_{mn}p_mq_n$$

The equation above is a weighted average, or ***expected payoff***, of the payments to Player 1. We indicate this expected payoff by $E(\mathrm{p}, \mathrm{q})$ to emphasize the fact that it depends on the strategies of the two players – we could express the expected payoff in matrix notation as follows

$$E(\mathrm{p},\mathrm{q}) = \begin{bmatrix} p_1 & p_2 & . & . & . & p_m \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{bmatrix}$$

e.g. if we use the carnival game that was shown earlier

$$E(p,q) = pAq = \begin{bmatrix} \tfrac{3}{8} & \tfrac{1}{4} & \tfrac{3}{8} \end{bmatrix} \begin{bmatrix} 2 & 0 & -1 & -2 \\ -3 & 6 & -3 & -4 \\ 4 & -5 & 5 & 8 \end{bmatrix} \begin{bmatrix} \tfrac{1}{4} \\ \tfrac{1}{8} \\ \tfrac{1}{8} \\ \tfrac{1}{2} \end{bmatrix} = 103/64 = 1.609$$

So in the long run, Player 1 can expect to get an average payment of about 1.61 from Player 2 in each play of the game.

But what about a game where both players are able to change their strategies independently of each other?

For this I will introduce the **Fundamental Theorem of Two-Person Zero-Sum Games**.

Fundamental Theorem of Zero-Sum Games

There exists strategies p^* and q^* such that

$$E(p^*, q) \geq E(p^*, q^*) \geq E(p, q^*)$$

For all strategies p and q

- p^* is an **optimal strategy for Player 1**
- q^* is an **optimal strategy for Player 2**
- $v = E(p^*, q^*)$ is called the **value** of the game

An entry a_{ij} in a payoff matrix A is called a **saddle point** if

- a_{ij} is the smallest entry in its row
- a_{ij} is the largest entry in its column

A game whose payoff matrix has a saddle point is called **strictly determined**

Examles of payoff matrices with a saddle point (in bold)

$$\begin{bmatrix} 2 & 0 & -1 & \mathbf{-2} \\ -3 & 6 & -3 & -4 \\ 4 & -5 & 5 & -8 \end{bmatrix} \qquad \begin{bmatrix} 44 & \mathbf{32} & 84 \\ 68 & 21 & 40 \\ 75 & 29 & 90 \end{bmatrix}$$

If a matrix has a saddle point a_{ij}, then the following strategies composed of the ith and jth entry are optimal strategies for the two players.

Strategies for which one move is possible are called **pure strategies**, while strategies with more that one move possible are called **mixed strategies**

A special case where optimal strategies can be found by elementary operations is when the payoff matrix is a 2 x 2 square matrix. If the game is strictly determined then the optimal strategy is easily found because a 2 x 2 matrix that is strictly determined must have at least one saddle point. If the game is not strictly determined then we use the following

Optimal Strategies for a 2 × 2 Matrix Game

For a 2 × 2 game that is not strictly determined, optimal strategies for players R C are

$$\mathbf{p}^* = \begin{bmatrix} \dfrac{a_{22} - a_{21}}{a_{11} + a_{22} - a_{12} - a_{21}} & \dfrac{a_{11} - a_{12}}{a_{11} + a_{22} - a_{12} - a_{21}} \end{bmatrix}$$

and

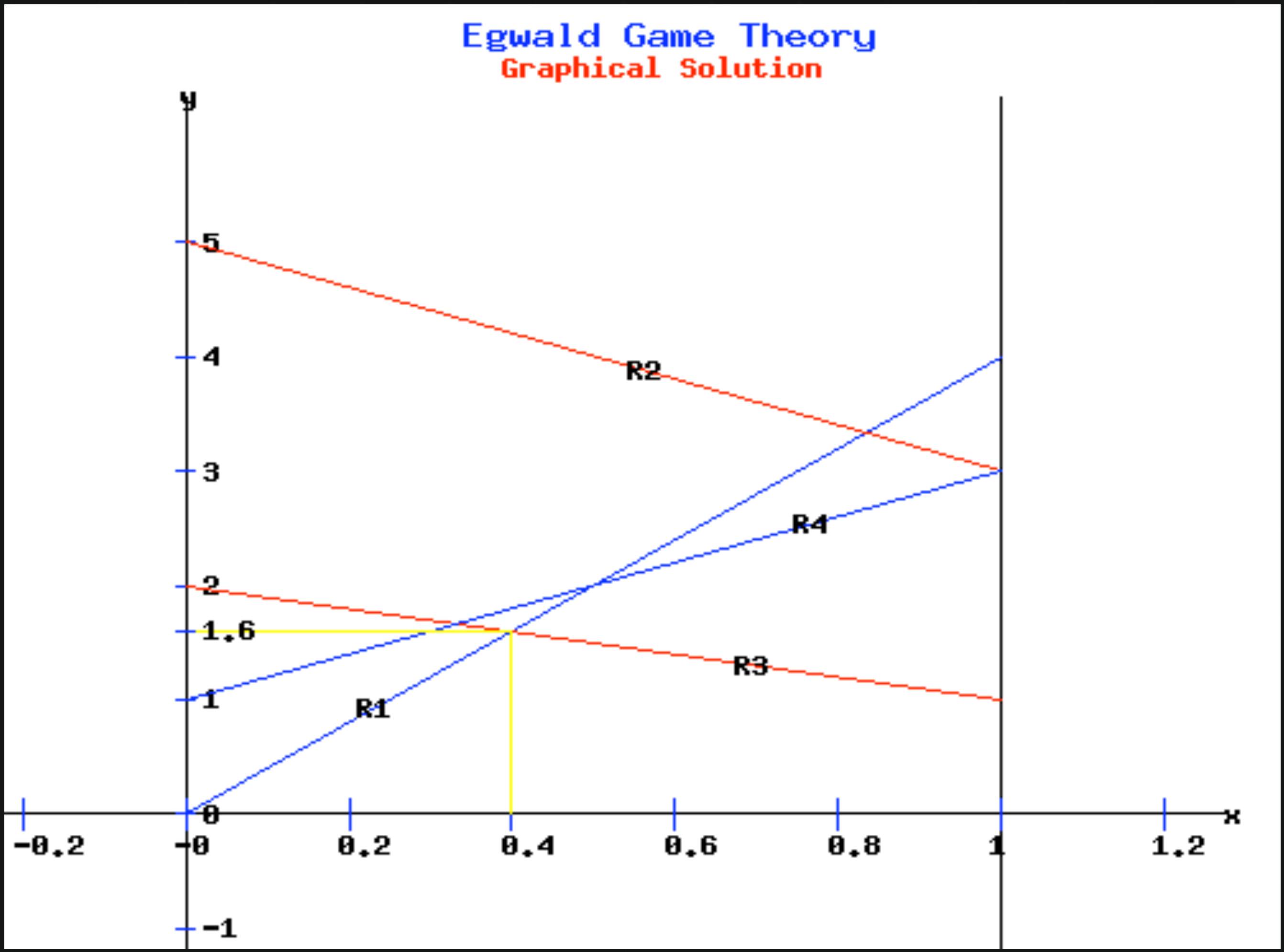
$$\mathbf{q}^* = \begin{bmatrix} \dfrac{a_{22} - a_{12}}{a_{11} + a_{22} - a_{12} - a_{21}} \\ \dfrac{a_{11} - a_{21}}{a_{11} + a_{22} - a_{12} - a_{21}} \end{bmatrix}$$

The value of the game is

$$v = \frac{a_{11}a_{22} - a_{12}a_{21}}{a_{11} + a_{22} - a_{12} - a_{21}}$$

* Replace Player R with Player 1
* Replace Player C with Player 2

The theory behind games of strategy doesn't just need to be used for "games", there are many uses for these theorems in other areas, such as: comparative advantage, a schedule for two different viewing networks, etc.



- A blank example for a geometric interpretation of the output for a particular 2x2 matrix game of strategy.

Acknowledgements

- *Elementary Linear Algebra with Applications* (Anton, Howard and Chris Rorres, 2005).
- *Operations Research - Game Theory* (Wiens, Elmer) <http://www.egwald.ca/operationsresearch/images/game_graph_0.php>