MATRIX DIAGONALIZATION & FIBONACCI NUMBERS

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INTRODUCTION

EIGENVALUES & EIGENVECTORS

An eigenvector of an nxn matrix A is an non-zero vector x: Ax = λx for some scalar λ. (called the associated eigenvalue of A).
→Give Anxn, λ = solution of equation

|A – λI | = 0

DIAGONALIZATION

A nxn is called **diagonalizable** if there exists an invertible matrix *P* of the same size satisfying $P^{-1}AP = D$ for some diagonal matrix D.

Compute power of A:

$$A = PDP^{-1}$$

$$A^2 = (PDP^{-1})(PDP^{-1}) = PD(P^{-1}P)DP^{-1} = PD(I)DP^{-1} = PD^2P^{-1}$$

$$A^3 = (PD^2P^{-1})(PDP^{-1}) = PD^2(P^{-1}P)DP^{-1} = PD^2(I)DP^{-1} = PD^3P^{-1}$$

$$\vdots$$

<u>**Theorem**</u> A *nxn* matrix & $\lambda_1,\ldots,\lambda_k$ be the distinct (real) eingenvalues of *A*. We have:

1. *A* is diagonalizable if and only if dim (E_{λ_1}) +…+ dim $(E_{\lambda_k})=n$.

2. If *A* is diagonalizable with $P^{-1}AP=D$, then the columns of *P* are basis vectors for the eigenspaces of *A* and the diagonal entries of *D* are the corresponding eigenvalues.

FIBONACCI NUMBER

The Fibonacci sequence (1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144 . . .) occurs throughout the worlds of nature, art, music, and mathematics!

Each term in the series is produced by adding together the two previous terms, so that 1 + 1=2, 1 + 2=3, 2 + 3=5, and so on.

$$f_n=f_{n-1}+f_{n-2}.$$

(from thirteenth-century Europe)

METHOD

STUDY OF A MATRIX A: $A = \begin{bmatrix}1 & 1\\ 1 & 0\end{bmatrix}$

$$|A-\lambda I| = 0 \to \begin{vmatrix}1-\lambda & 1\\ 1 & -\lambda\end{vmatrix} = 0 \Rightarrow \lambda^2-\lambda-1 = 0. \to \lambda_1 = \frac{1+\sqrt{5}}{2},\quad \lambda_2 = \frac{1-\sqrt{5}}{2}.$$

$\lambda_1=(1+\sqrt{5})/2$:

$$\left[\begin{array}{cc|c}\frac{(1-\sqrt{5})}{2} & 1 & 0\\ 1 & -\frac{(1+\sqrt{5})}{2} & 0\end{array}\right] \to \left[\begin{array}{cc|c}1 & -\frac{(1+\sqrt{5})}{2} & 0\\ 0 & 0 & 0\end{array}\right] \to X_1 = \begin{bmatrix}\frac{(1+\sqrt{5})}{2}\\ 1\end{bmatrix};\quad X_2 = \begin{bmatrix}\frac{(1-\sqrt{5})}{2}\\ 1\end{bmatrix}$$

General solution: X1 Similarly X2:

From theorem:

1. **dim (E_{λ_1}) + dim (E_{λ_2})=2**
2. → *A* is diagonalizable

$$P = \begin{bmatrix}\frac{(1+\sqrt{5})}{2} & \frac{(1-\sqrt{5})}{2}\\ 1 & 1\end{bmatrix} \to P^{-1} = \begin{bmatrix}\frac{1}{\sqrt{5}} & \frac{-(1-\sqrt{5})}{2\sqrt{5}}\\ \frac{-1}{\sqrt{5}} & \frac{(1+\sqrt{5})}{2\sqrt{5}}\end{bmatrix}.$$

$$\to A = \begin{bmatrix}\frac{(1+\sqrt{5})}{2} & \frac{(1-\sqrt{5})}{2}\\ 1 & 1\end{bmatrix}\begin{bmatrix}\frac{(1+\sqrt{5})}{2} & 0\\ 0 & \frac{(1-\sqrt{5})}{2}\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{5}} & \frac{-(1-\sqrt{5})}{2\sqrt{5}}\\ \frac{-1}{\sqrt{5}} & \frac{(1+\sqrt{5})}{2\sqrt{5}}\end{bmatrix}$$

Applied to Fibonacci sequence: $f_n=f_{n-1}+f_{n-2}$.

Let: $V_n = \begin{bmatrix}f_{n+1}\\ f_n\end{bmatrix} \to V_{n+1} = \begin{bmatrix}f_{n+2}\\ f_{n+1}\end{bmatrix} = \begin{bmatrix}1 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}f_{n+1}\\ f_n\end{bmatrix} = AV_n$; where $A = \begin{bmatrix}1 & 1\\ 1 & 0\end{bmatrix}$

$\to V_n = AV_{n-1},\ . n \geq 0$ We have: $V_n = AV_{n-1} = A^2V_{n-2} = A^3V_{n-3} = \ldots = A^nV_0.$

But $A^n=PD^nP^{-1}$ (as studied above)

$$\to V_n = PD^nP^{-1}V_0 = \begin{bmatrix}\frac{(1+\sqrt{5})}{2} & \frac{(1-\sqrt{5})}{2}\\ 1 & 1\end{bmatrix}\begin{bmatrix}\left(\frac{1+\sqrt{5}}{2}\right)^n & 0\\ 0 & \left(\frac{1-\sqrt{5}}{2}\right)^n\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{5}} & \frac{-(1-\sqrt{5})}{2\sqrt{5}}\\ \frac{-1}{2\sqrt{5}} & \frac{(1+\sqrt{5})}{2\sqrt{5}}\end{bmatrix}\begin{bmatrix}1\\ 1\end{bmatrix}$$

carefully multiply out → the n^{th} Fibonacci term:

$$f_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right]$$

e.g. for n = 100:

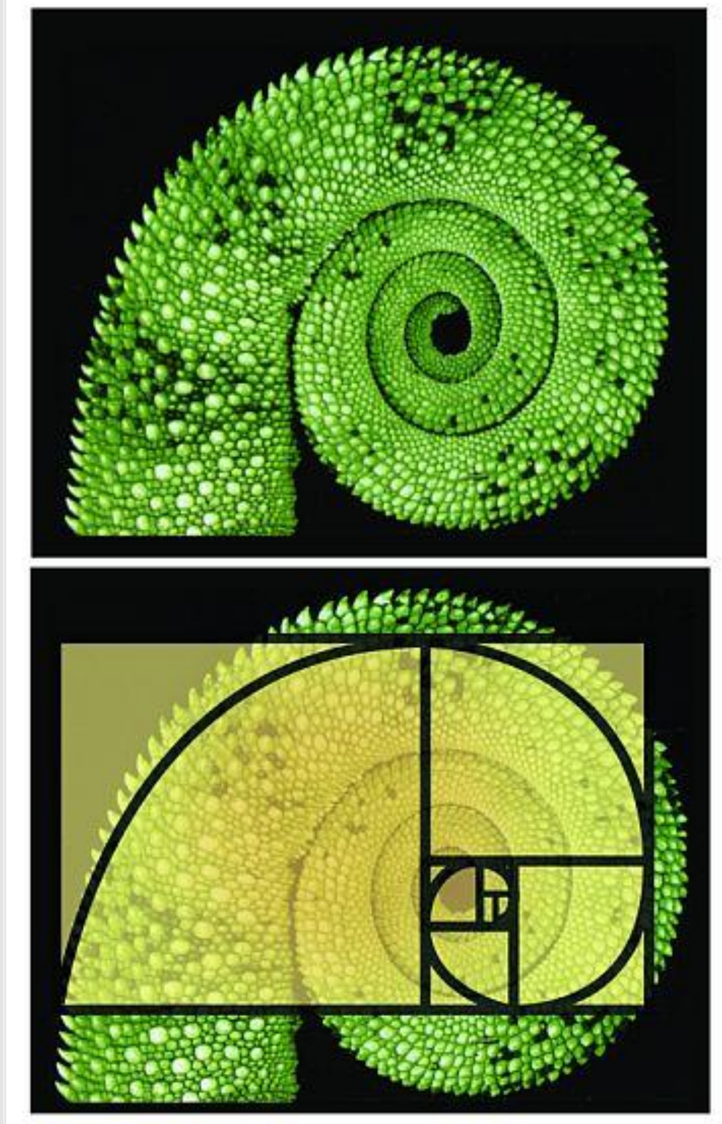
$$f_{100} = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{101} - \left(\frac{1-\sqrt{5}}{2}\right)^{101}\right]$$

= 573147844013817084101.

SUMMARY

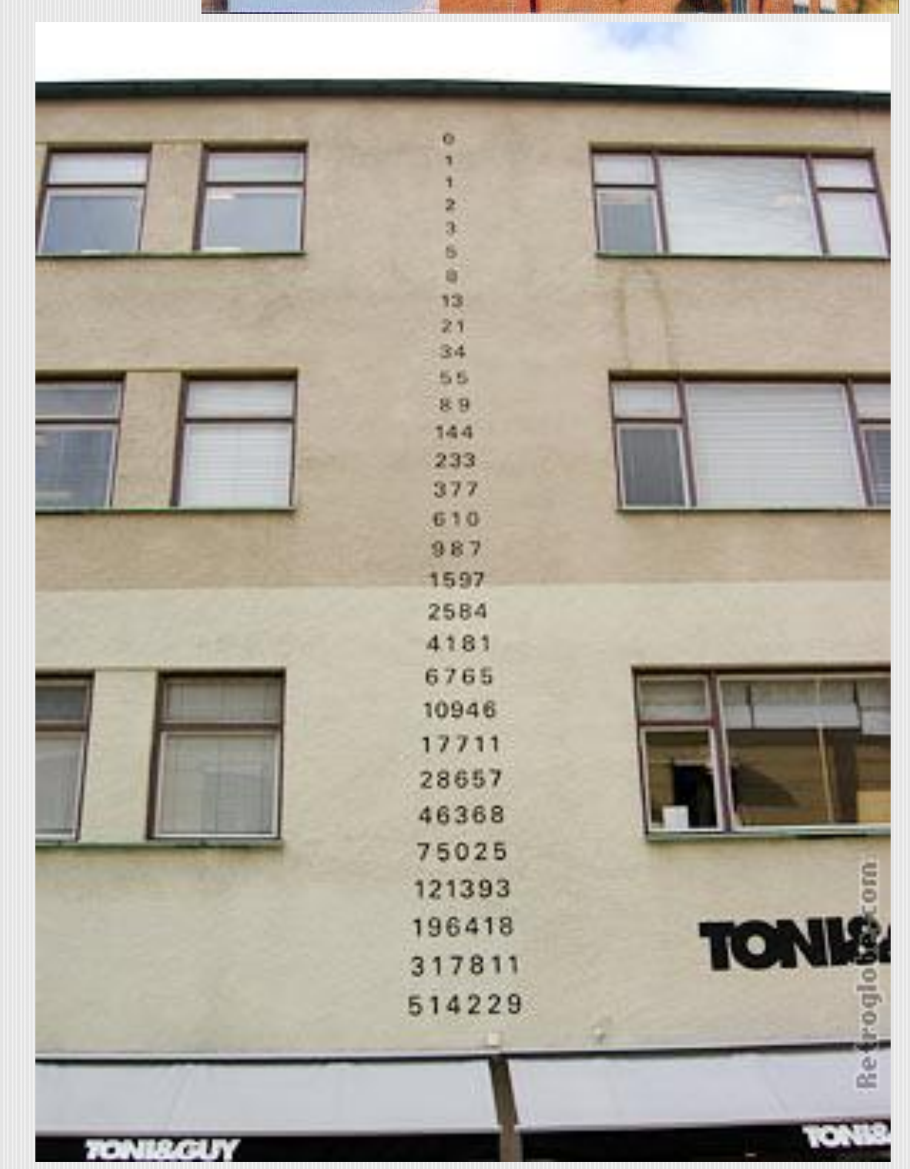
The Fibonacci numbers get bigger and bigger. The expression that gives *fn* for any *n* can be found if one knows a little bit about matrix diagonalization.

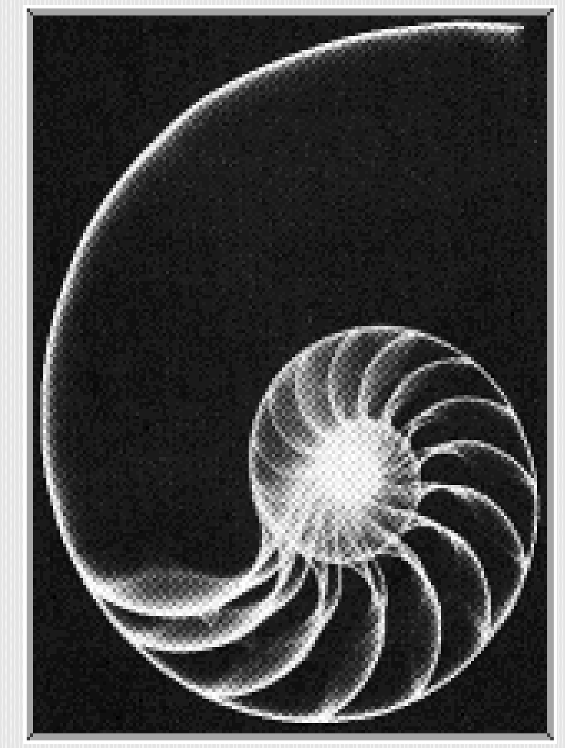
Examples of FIBONACCI ART

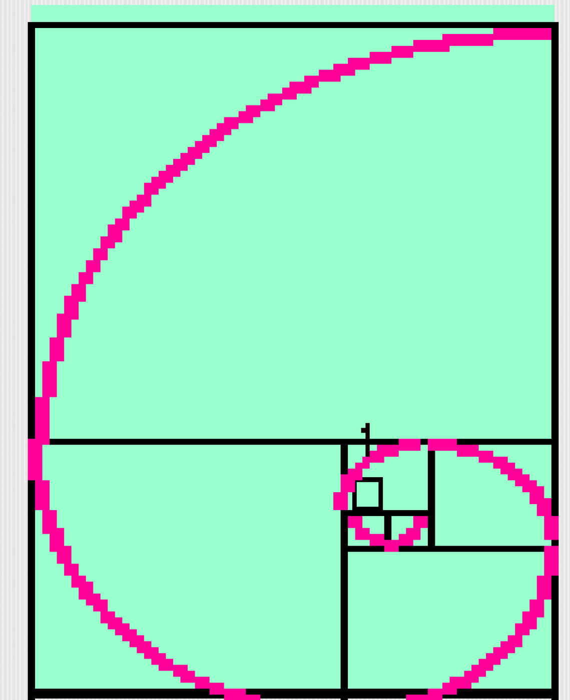


Chameleon Tail - Fibonacci Pattern

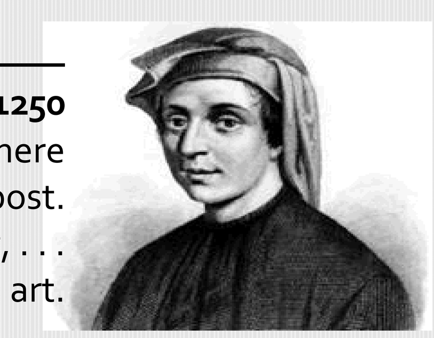
Fibonacci numbers in smokestack (Turku) and building (Sweden).







Spiral shells also exhibit pattern:



Fibonacci 1170-1250
Leonardo Pisano, known by his nick name Fibonacci, was born in Pisa, Italy, but was educated in North Africa (Algeria) where his father, Guilielmo, held a diplomatic post.
Fibonacci numbers: 1, 1, 2, 3, 5, 8, 13, 21, 55, 89, 144, 233, 377, . . .
This sequence occurs in many places in math, general science, in nature and even in art.