Connectivity Matrix (also known as Adjacency Matrix) and its applications to Airline Flights

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INTRODUCTION

An airline serves five cities, say, A, B, C, D, and H, in which H is the "hub city." The various routes between the cities are indicated in Figure 3.5.1.

Suppose you wish to travel from city A to city B so that at least two connecting flights are required to make the trip. Flights $(A \to H)$ and $(H \to B)$ provide the minimal number of connections. However, if space on either of these two flights is not available, you will have to make at least three flights. Several questions arise. How many routes from city A to city B require *exactly* three connecting flights? How many routes require *no more than* four flights—and so forth? Since this particular network is small, these questions can be answered by "eyeballing" the diagram, but the "eyeball method" won't get you very far with the large networks that occur in more practical situations. Let's see how matrix algebra can be applied. Begin by creating a ***connectivity matrix*** $\mathbf{C} = [c_{ij}]$ (also known as an ***adjacency matrix***) in which

$$c_{ij} = \begin{cases} 1 & \text{if there is a flight from city } i \text{ to city } j, \\ 0 & \text{otherwise.} \end{cases}$$

PROPOSED METHOD

For the network depicted in Figure 3.5.1,

$$\mathbf{C} = \begin{array}{c} \\ A \\ B \\ C \\ D \\ H \end{array} \begin{array}{c} \begin{array}{ccccc} A & B & C & D & H \end{array} \\ \begin{pmatrix} 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix} \end{array}.$$

The matrix $\mathbf{C}$ together with its powers $\mathbf{C}^2, \mathbf{C}^3, \mathbf{C}^4, \ldots$ will provide all of the information needed to analyze the network. To see how, notice that since c_{ik} is the number of direct routes from city i to city k, and since c_{kj} is the number of direct routes from city k to city j, it follows that $c_{ik}c_{kj}$ must be the number of 2-flight routes from city i to city j that have a connection at city k. Consequently, the (i,j)-entry in the product $\mathbf{C}^2 = \mathbf{CC}$ is

$$[\mathbf{C}^2]_{ij} = \sum_{k=1}^{5} c_{ik}c_{kj} = \text{ the total number of 2-flight routes from city } i \text{ to city } j.$$

Similarly, the (i,j)-entry in the product $\mathbf{C}^3 = \mathbf{CCC}$ is

$$[\mathbf{C}^3]_{ij} = \sum_{k_1,k_2=1}^{5} c_{ik_1}c_{k_1k_2}c_{k_2j} = \text{ number of 3-flight routes from city } i \text{ to city } j,$$

and, in general,

$$[\mathbf{C}^n]_{ij} = \sum_{k_1,k_2,\cdots,k_{n-1}=1}^{5} c_{ik_1}c_{k_1k_2}\cdots c_{k_{n-2}k_{n-1}}c_{k_{n-1}j}$$

is the total number of n-flight routes from city i to city j. Therefore, the total number of routes from city i to city j that require *no more* than n flights must be given by

$$[\mathbf{C}]_{ij} + [\mathbf{C}^2]_{ij} + [\mathbf{C}^3]_{ij} + \cdots + [\mathbf{C}^n]_{ij} = [\mathbf{C} + \mathbf{C}^2 + \mathbf{C}^3 + \cdots + \mathbf{C}^n]_{ij}.$$

Connecting Flights by eyeballing.

LET,
A: Manhattan, New York
B: Chicago, Illinois
C: Miami, Florida
D: Los Angeles, California
H: Dallas, Texas

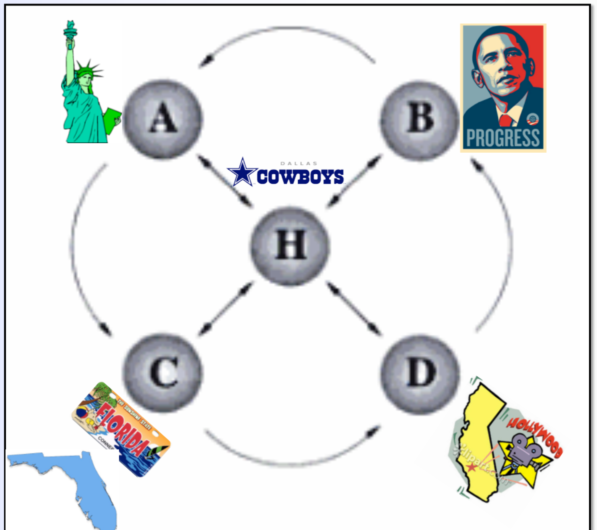


FIGURE 3.5.1

RESULTS

For our particular network,

$$\mathbf{C}^2 = \begin{pmatrix} 1 & 1 & 1 & 2 & 1 \\ 1 & 1 & 2 & 1 & 1 \\ 1 & 2 & 1 & 1 & 1 \\ 2 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 4 \end{pmatrix}, \ \mathbf{C}^3 = \begin{pmatrix} 2 & 3 & 2 & 2 & 5 \\ 2 & 2 & 2 & 3 & 5 \\ 3 & 2 & 2 & 2 & 5 \\ 2 & 2 & 3 & 2 & 5 \\ 5 & 5 & 5 & 5 & 4 \end{pmatrix}, \ \mathbf{C}^4 = \begin{pmatrix} 8 & 7 & 7 & 7 & 9 \\ 7 & 8 & 7 & 7 & 9 \\ 7 & 7 & 8 & 7 & 9 \\ 7 & 7 & 7 & 8 & 9 \\ 9 & 9 & 9 & 9 & 20 \end{pmatrix},$$

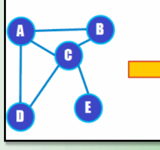
and

$$\mathbf{C} + \mathbf{C}^2 + \mathbf{C}^3 + \mathbf{C}^4 = \begin{pmatrix} 11 & 11 & 11 & 11 & 16 \\ 11 & 11 & 11 & 11 & 16 \\ 11 & 11 & 11 & 11 & 16 \\ 11 & 11 & 11 & 11 & 16 \\ 16 & 16 & 16 & 16 & 28 \end{pmatrix}.$$

The fact that $[\mathbf{C}^3]_{12} = 3$ means there are exactly 3 three-flight routes from city A to city B, and $[\mathbf{C}^4]_{12} = 7$ means there are exactly 7 four-flight routes—try to identify them. Furthermore, $[\mathbf{C} + \mathbf{C}^2 + \mathbf{C}^3 + \mathbf{C}^4]_{12} = 11$ means there are 11 routes from city A to city B that require no more than 4 flights.

DISCUSSIONS

Network



Connectivity Matrix

	A	B	C	D	E
A	0	1	1	1	0
B	1	0	1	0	0
C	1	1	0	1	1
D	1	0	1	0	0
E	0	0	1	0	0

Connectivity Matrix

The network on the above figure can be represented as a connectivity matrix, which is rather simple to construct:

Size of the connectivity matrix: involves a number of rows and cells equivalent to the number of nodes in the network. Since the above network has 5 nodes, its connectivity matrix is a five by five grid.

Connection: Each cell representing a connection between two nodes receives a value of 1 (e.g. Cell B - A).

Non-connection: Each cell that does not represent a direct connection gets a value of 0 (e.g. Cell D - E)

.If all connections in the network are bi-directional (a movement is possible from node C to node D and vice-versa), the connectivity matrix is **transposable.**

Adding up a row or a column gives the **degree of a node**. Node C is obviously to most connected since it has the highest summation of connectivity comparatively to all other nodes. However, this assumption may not hold true on a more complex network because of a larger number of indirect paths which are not considered in the connectivity matrix. For example, when connecting hundreds of flights to fit schedules, liking cities together, and other factors requires a further complex network that involves deeper analysis. That sort of job is left to the professionals. For our particular network, we examined one of the most basic forms of connectivity matrix and its application to airline flights.

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