An Academic Journey of “Tangent”

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Cypress College Math Club
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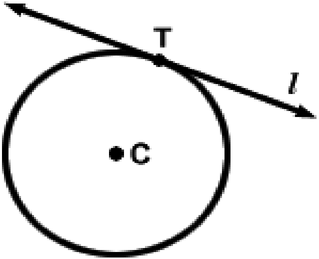
Outline

1. High School
   - Geometry Class
   - Algebra Class
2. College
   - Calculus I
   - Calculus II
   - Calculus III
3. Graduate School
   - Pure Math
   - Applied Math
4. Real Life
5. Conclusions

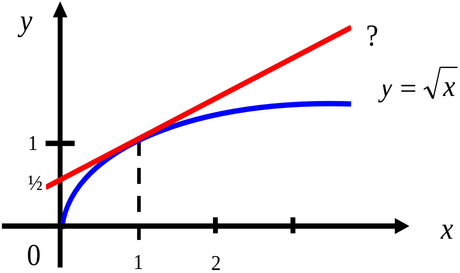
Geometrically

What is a Tangent line geometrically?



Algebraically

What is the equation of a Tangent line and why?



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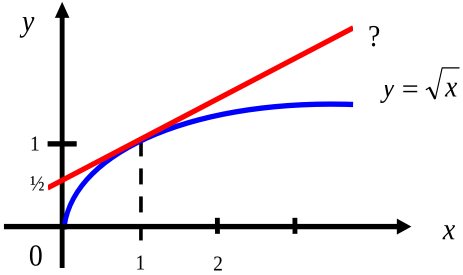

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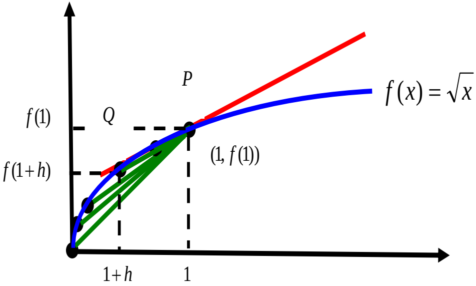
Average R.O.C. and Secant



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2. But what does $\dfrac{f(1+h)-f(1)}{(1+h)-1}$ also represent? Slope of the **secant** lines

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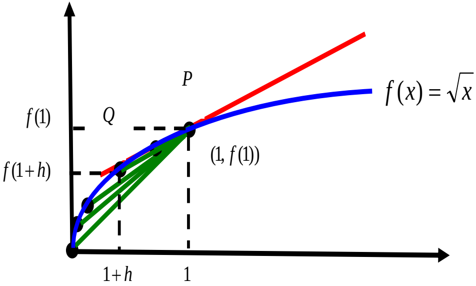

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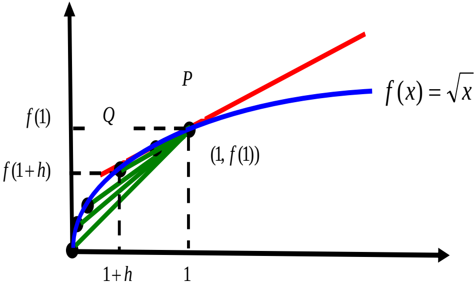

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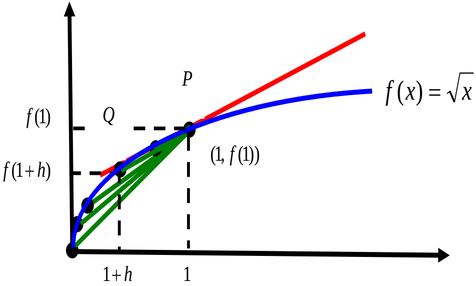

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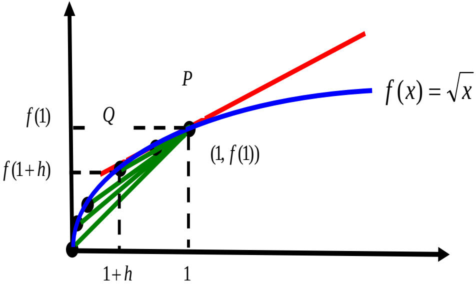

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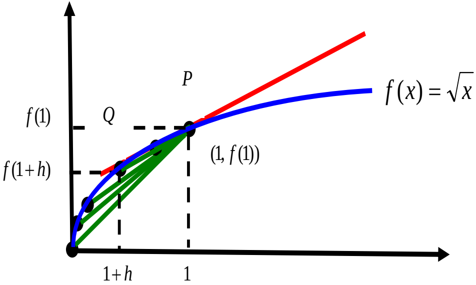

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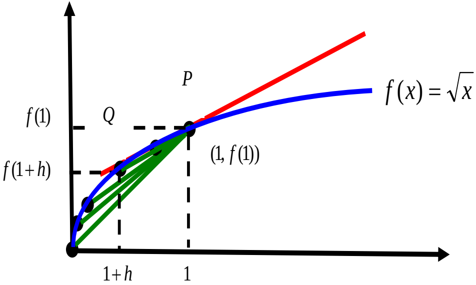

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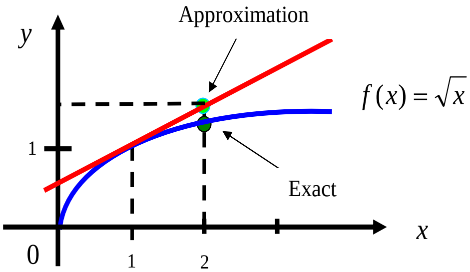
First Degree Approximation to Curves

Question: What is $\sqrt{2}$?

Geometry of Tangent Approximation:



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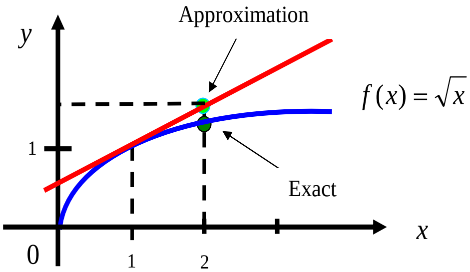

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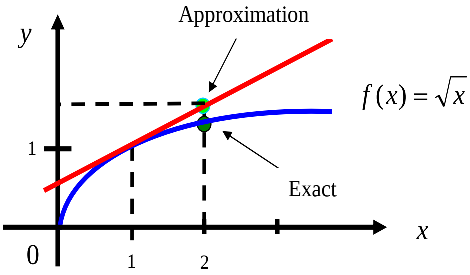

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Taylor Approximation

English mathematician Brook Taylor came up with a way to approximate functions $f(x)$ at a point $x = a$ using polynomials:

$$\begin{aligned} f(x) &= \sum_{n=1}^{\infty} \frac{f^{(n)(a)}}{n!}(x-a)^n \\ &= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots \end{aligned}$$

The truncation

$$f(x) \approx f(a) + f'(a)(x-a)$$

is called the Tangent approximation.

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Taylor Approximation for $\sqrt{2}$

So for $f(x) = \sqrt{x}$ with $a = 1$, we can approximate $\sqrt{2}$ using the series

$$\begin{aligned}\sqrt{x} &\approx 1 + \frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2 + \frac{1}{16}(x-1)^3 \\ &- \frac{5}{128}(x-1)^4 + \frac{7}{256}(x-1)^5 - + \cdots\end{aligned}$$

Now, for $x = 2$

$$\sqrt{2} \approx 1 + \frac{1}{2} - \frac{1}{8} + \frac{1}{16} - \frac{5}{128} + \frac{7}{256} - \frac{21}{1024} + \frac{33}{2048} - \frac{429}{32768} + - \cdots$$

According to Texas Instruments: "*most of our graphing calculators use the common Taylor Series to calculate some functions. This is especially true for calculators with CAS, such as the TI-89 Titanium and TI-Nspire CAS.*"

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Calculate $\sqrt{2}$ By Hand

If you wonder how people calculated square roots in the stone ages (time before scientific calculators, slide rules, and square root tables):

$$\sqrt{2.000000}$$

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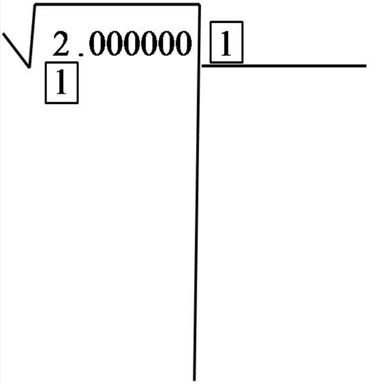

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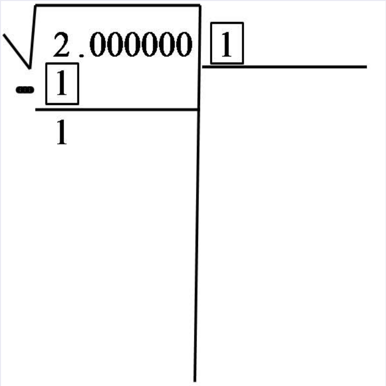

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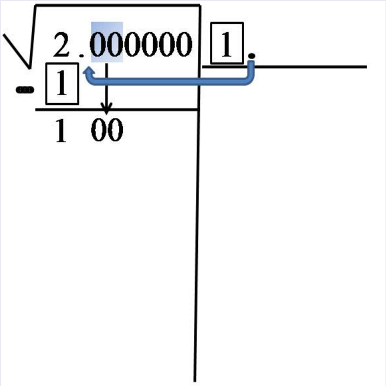

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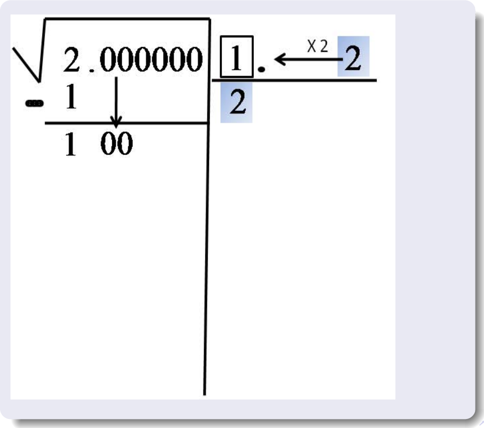

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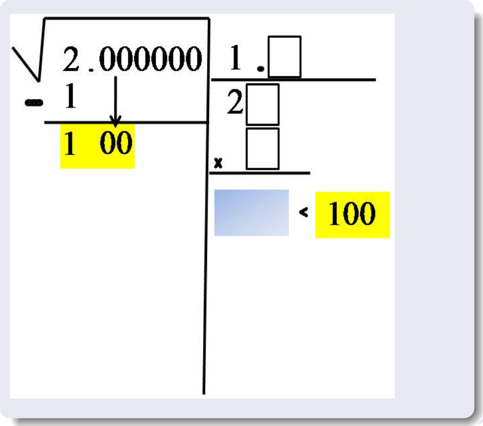

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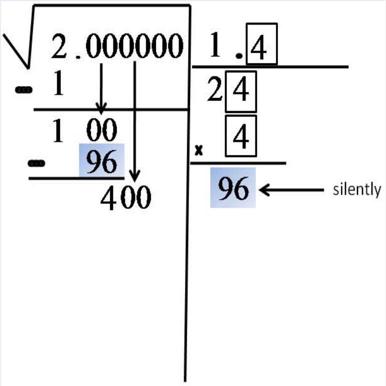

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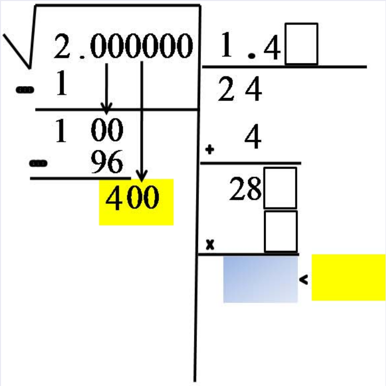

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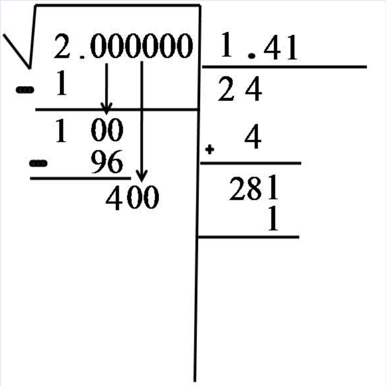

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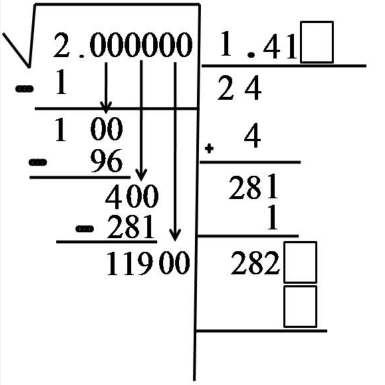

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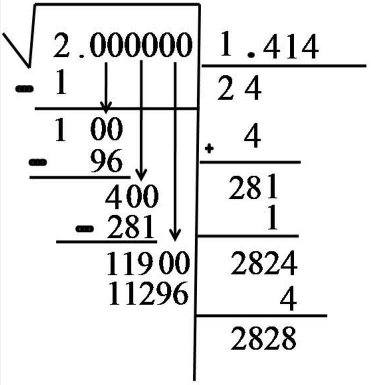

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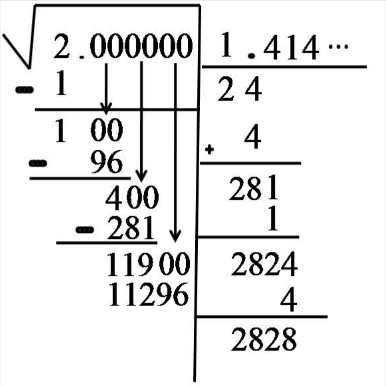
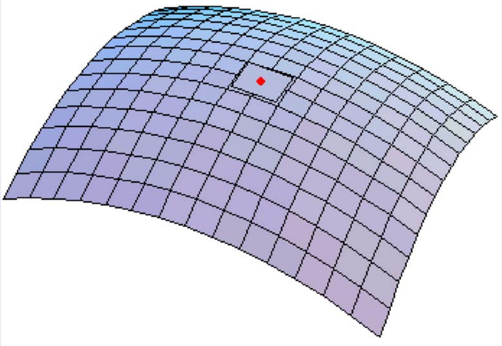


Tangent Plane

The idea of Tangent approximations can be generalized in higher dimensions. For example, a 2-dimensional Tangent plane is a linear approximation to a 3-dimensional surface.



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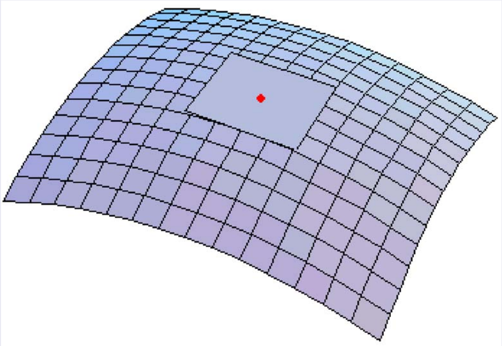

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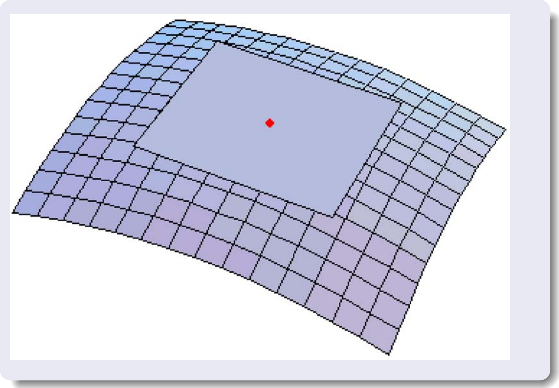

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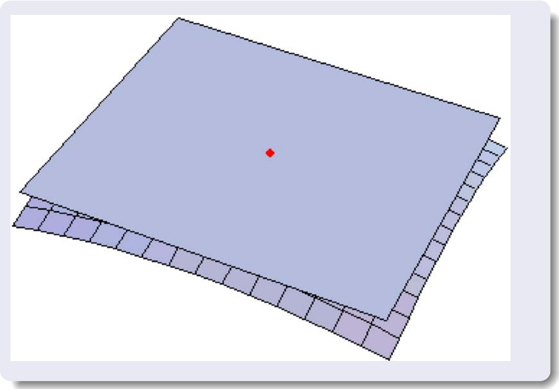

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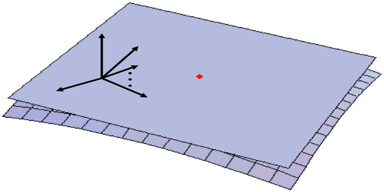
Tangent Space

- Similarly, we can use a Tangent space to approximate a high-dimensional surface (e.g., a manifold).



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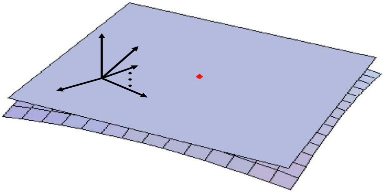

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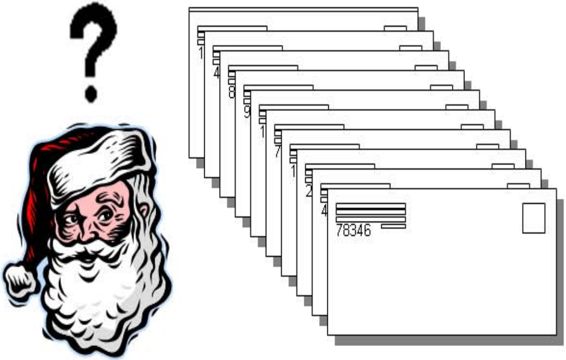
Handwritten Digit Classification

Santa thought to himself, “only if these mails can go to the right place according to their zip code”.



Handwritten Digit Classification

Problem. (Human) handwritten digits are sometimes very hard to recognize, even by human operators.

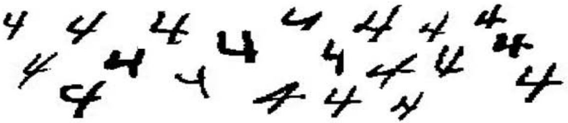
Importance. Accurate identification of the digits ensures a reliable delivery system.

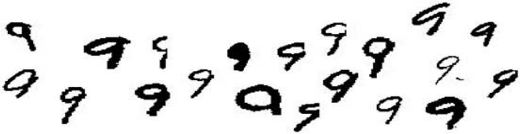
Beneficiaries. Postal services (mail sorting), seaports (cargo registration), etc.

Even Santa Clause can benefit from an efficient digit classification algorithm.

Problem Definition

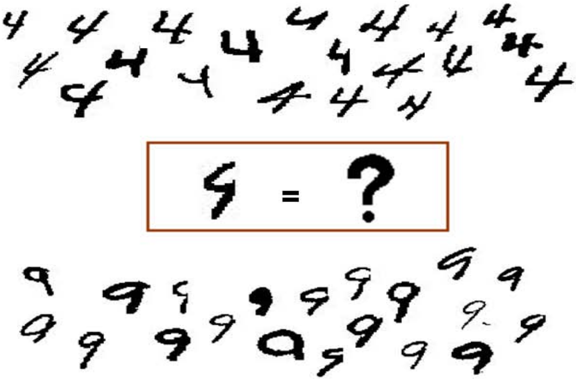
How do we tell a bunch of 4's from a bunch of 9's?





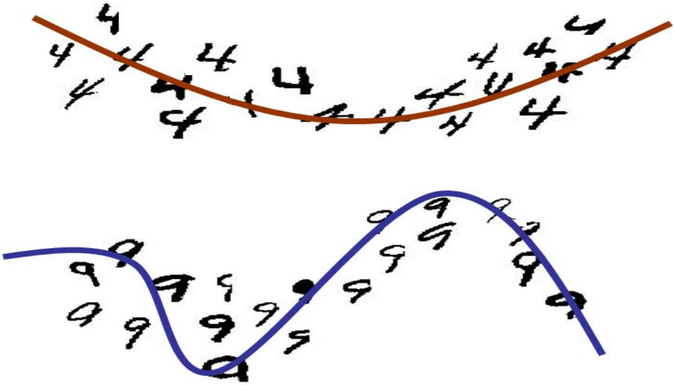
Problem Definition

Or, how do we tell whether a new digit is a 4 or a 9?



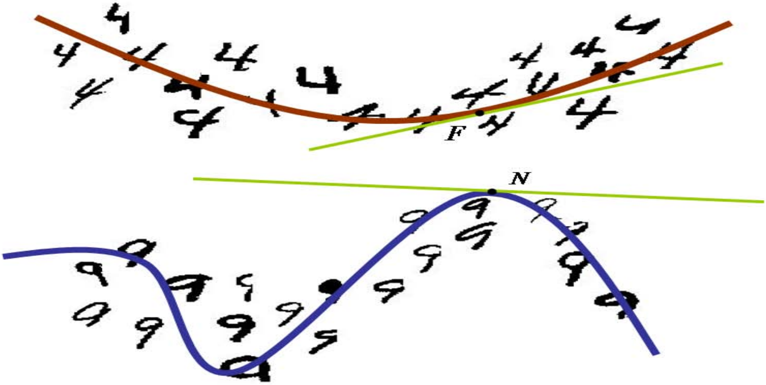
Digit Manifolds

Imagine a high-D surface (red curve) where all 4's live on and a high-D surface (blue curve) where all 9's live on.

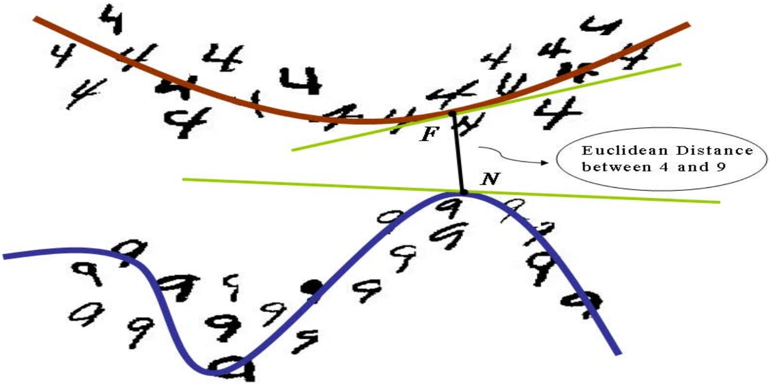


Tangent Spaces - Training

Create a Tangent Space of the 4's at F and create a Tangent Space of the 9's at N.



Euclidean Distance



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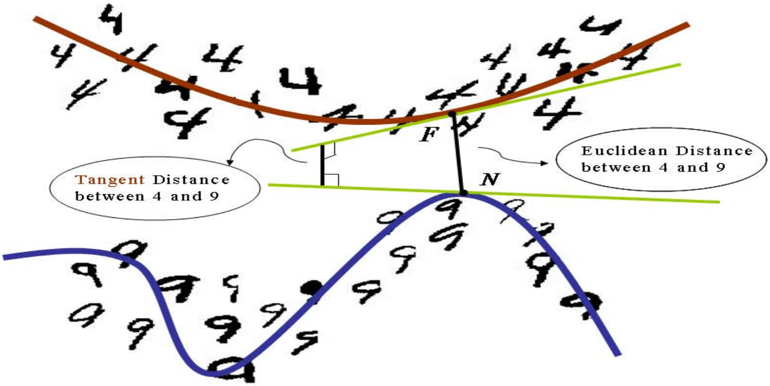
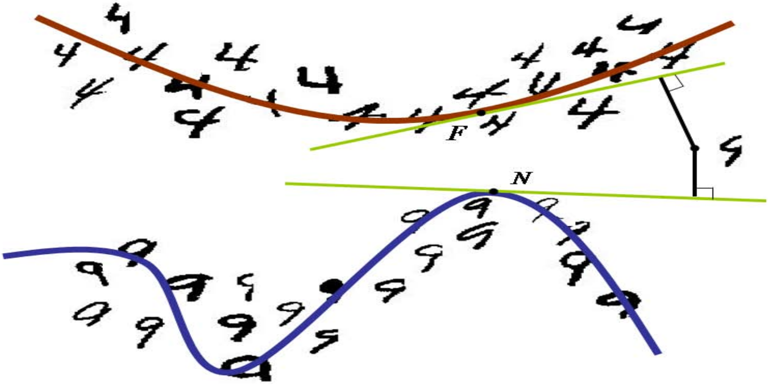
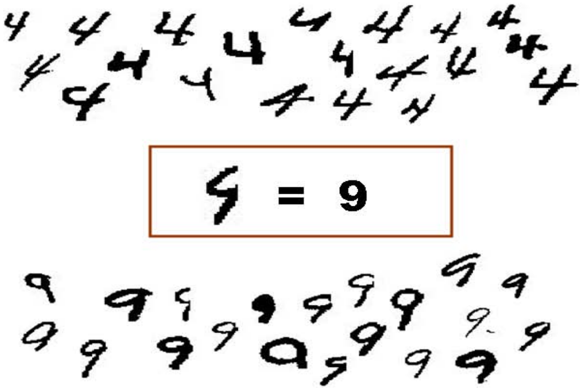


Classification

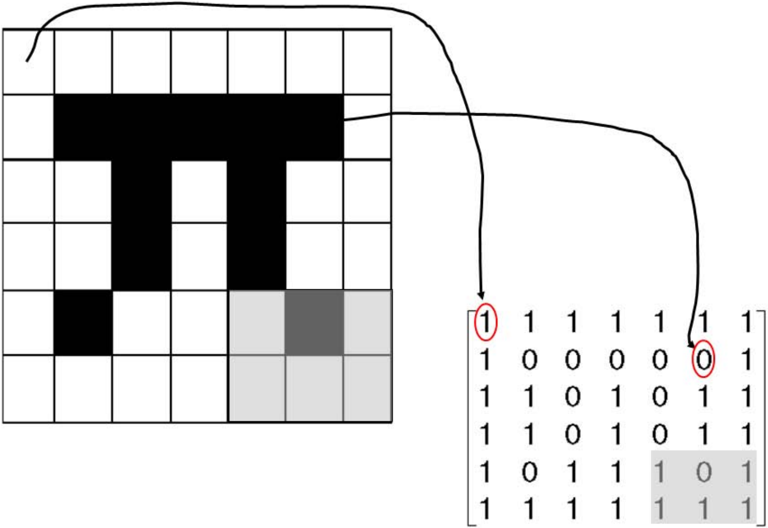
So, is it a 4 or a 9?



Classification Result



Face Recognition

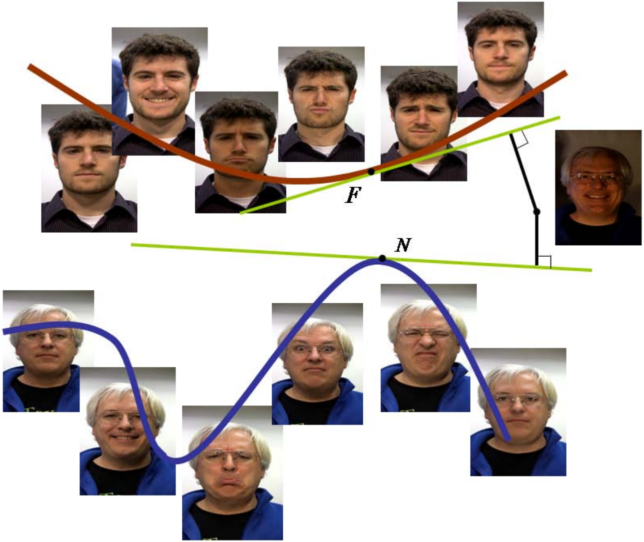


Face Recognition

$$
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 1 & 0 & 1 & 0 & 1 & 1 \\
1 & 1 & 0 & 1 & 0 & 1 & 1 \\
1 & 0 & 1 & 1 & 1 & 0 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1
\end{bmatrix}
\longrightarrow
\begin{bmatrix}
1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ \hline 1 \\ 0 \\ 1 \\ 1 \\ 0 \\ 1 \\ \hline \vdots \\ \hline 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1
\end{bmatrix}
$$

IMAGE $\longrightarrow$ MATRIX $\longrightarrow$ VECTOR

Face Recognition



So, what did we learn from this?

1. Mathematics is cool and mathematicians rock.
2. Mathematics is the foundation in answering most scientific questions and we can not live without it.

The bottom line is that I hope I have helped answering the age-old question,

“Why I have to learn the math I learn in high school”

NOW, GO PICK A MATHEMATICAL CONCEPT AND SEE WHERE IT LEADS YOU ...

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The bottom line is that I hope I have helped answering the age-old question,

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