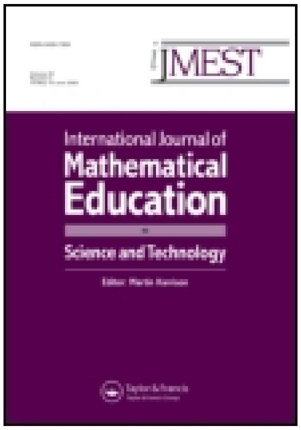

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A practical approach to inquiry-based learning in linear algebra

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2. A preliminary form of this argument was announced in a conference organized by the University of Crete in 2002 [11].
3. The line-segment AF called symmedian because the angle FAB is equal to angle MAC; see Figure 4. For more details on this term, see [2, Chap. 7]. For a proof of the equality of these angles, see [8, pp. 369–370].
4. In Figure 4, we observe that the points Z1, Z and Z2 are the reflections of F in the sides AI, IK and AK of the AIK triangle, respectively. The three feet N, S and O lie on the line NO which is called the Simson line of the point F (see also [2, Chap. 5]).

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A practical approach to inquiry-based learning in linear algebra

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Linear algebra has become one of the most useful fields of mathematics since last decade, yet students still have trouble seeing the connection between some of the abstract concepts and real-world applications. In this

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article, we propose the use of thought-provoking questions in lesson designs to allow two-way communications between instructors and students as well as the use of simple applications to motivate learning and keep students engaged. An illustrative lesson for learning matrix multiplication and a lesson for understanding the practical meaning of orthogonal projection that are motivated by applications in image processing are presented along with a ready-to-use MATLAB code for completeness.

Keywords: linear algebra; inquiry-based learning; image processing; applications; college-level; feature extraction; matrix multiplications; orthogonal projection

1. Introduction

Linear algebra was not treated as a standard mathematical subject until the 1930s [1]. When Van Der Waerden [2] and Birkhoff and MacLane [3] published their texts on modern algebra which included chapters on linear algebra, educators started to teach linear algebra with an axiomatic approach. It is not until the 1950s and 1960s that linear algebra became a separate and standard course in the college mathematics curriculum in the United States.

Although educators have successfully used the standard linear algebra course as a transition to abstract mathematics and introduction to proof, much of this mentality was under scrutiny as computers became widely available. Those complicated or tedious tasks in linear algebra, such as solving a linear system of 100 equations with 100 unknowns, can now be easily accomplished by computers. It is, indeed, this transition to the increasing reliance of technology that drove the linear algebra reform in the 1990s.

Over time, we have witnessed a gradual shift from the axiomatic treatment of linear algebra topics into a matrix-oriented approach [4]. This is again, heavily influenced by the development of array-like representation of computer memories as well as students' need for concreteness. Manipulations of matrices not only allow students to transcend from procedural learning to building effective concept images [5] of difficult concepts but also provide venues for validating abstraction.

As linear algebra curricula in the country are being designed to adapt to a technology- and application-driven culture, there has been a growing concern in the past three decades about the quality of this movement (see, e.g. [6–8]). Calculus reform in the 1980s brought the attention to educational reforms in linear algebra, among numerous others (e.g. differential equations and geometry). The peak of the debates occurred in the 1990s when the Linear Algebra Curriculum Study Group (LACSG) released a set of recommendations for the first course in linear algebra. The recommendations were formed based on research-based knowledge on how mathematics is learned and should be taught, and what pedagogical and epistemological considerations are involved in the learning and teaching of linear algebra [9]. The recommendations were also influenced by the individual teaching experiments of members of the LACSG and inputs from a variety of client disciplines.

While each linear algebra instructor still experiments and invents novel teaching strategies on a daily basis, there are some fundamental issues that hinder the potential success of the reform. For example, one of the difficulties educators encounter in moving forward with the linear algebra reform is magnified by students' lack of preparation of linear algebra concepts in early stages. In particular, high school curricula in the United States generally place higher emphasis on calculusrelated topics as a possible consequence of the advanced placement (AP) examination in calculus [10]. Furthermore, the lack of time allotted for linear algebra, students' background and readiness in regard to content (objects, language and ideas that are unique to linear algebra) and student's readiness in concept of proof all contribute to potential student failure in their first linear algebra course in college [1,7,10]. That being said, for the time we do get to teach concepts of linear algebra, we need to make it count and worthwhile for the students as early as high school.

2. Rationale to inquiry-based learning

We believe that the fundamental issue to be studied is not merely how to present materials better; rather, it is ultimately how students learn and perceive concepts in linear algebra. We are not proposing another innovative way of teaching linear algebra; rather, we propose to provide a systematic framework for instructors to become better listeners, to ask thought-provoking questions, to design lessons that facilitate conceptual understanding of key concepts in linear algebra, to help students make mental constructions of mathematical objects and to create a lasting effect in student learning of mathematics in general.

Traditionally, teaching is viewed as showing students clearly what we want them to know and not necessarily knowing why students do not perform well on the things we want them to learn. Such a view ignores the cognitive development that can be necessary even for so simple an idea as arranging tiles into rectangular arrays [6]. Repeatedly showing students what we want them to know will not automatically help students to translate the knowledge into their own. And simply doing a great job of telling and showing difficult concepts, such as linear independence, may not significantly improve student learning of such topics, either.

So, what should we do? A general consensus among mathematics education researchers at the Park City Mathematics Institute in 1998 (PCMI) suggests that we should become better *listeners* [11] – *Not only should we develop a systematic approach to tackle student misconceptions*, *but it is equally important*, *if not more*, *to better understand what the misconceptions are and how they develop*. Hence, we bear in mind two fundamental elements in our lesson designs – (1) promoting conceptual understanding of key topics through a series of well-directed questioning to really probe student thinking and detect their misconceptions and (2) adopting a motivation-first, theory-second approach to stimulate students' intellectual needs for learning. Specifically, we wish to gain from this practice the source of students' faulty ways of thinking, such as arriving at conclusions on the sole basis of assumptions without examining their meaning and truth. Through a series of well-posed questions, we can explore the steps students take to form a solid concept image – *a mental picture consisting of what the person knows about the concept* (*e.g. similarity and difference to other concepts*, *examples and non-examples*, *etc.*) [5].

3. Rationale to a practical approach

From my personal experience in learning and teaching linear algebra over the years, I have noticed that failure in linear algebra is often associated with the fact that students fail to see the connection of seemingly abstract topics to real-world applications. Students may have an easier time with manipulations of matrices than understanding the purpose of learning vector spaces and inner product spaces; however, they have a hard time finding uses of all abstract mathematical ideas in daily life. I find myself constantly looking for ways to convince students what they are learning is worthwhile and useful, and motivate difficult concepts with interesting applications that students can relate to. I do so *before* a concept is formally introduced instead of giving it *after* as an application. This approach helps to keep students engaged and provides a concrete medium for students to relate to during the lesson.

We understand that people learn new things differently. Some are visual, some are auditory and some are both. In fact, approximately 20–30% of the school-aged population remembers what is heard; 40% remembers well visually the things that were seen or read; many must write or use their fingers in some manipulative way to help them remember basic facts; others cannot internally convert information or skills unless they use them in real-life activities. We believe that this aspect should not be overlooked in a well-rounded lesson design. It might not be possible to design lessons that incorporate all three aspects at once all the time due to constraints such as the location of the classroom and availability of the hardware and software. In general, instructors should strive to find what works best for their teaching style in consideration of the resources available.

For the remainder of this article, we present two example lessons that supply a set of discussion topics and incorporate simple image processing applications to offer a practical purpose to learning matrix multiplication and orthogonal projection. Instructors may use this guide to accomplish two things simultaneously – (1) by soliciting answers from students, faculty listeners have a chance to correct students' faulty ways of thinking and (2) by posing questions appropriately, students are expected to be actively involved in the problem-solving process instead of passively receiving information from instructors.

4. Matrix multiplication lesson

In this example, we give an example lesson on matrix multiplication through a practical use of feature extraction. The lesson presented here requires minimally an overhead projector and can be easily transformed into a computer lab activity if desired. Instructors who have access to a smart classroom might find it even more beneficial to illustrate the example with a computer and LCD projector for better resolution.

Objectives:

(1) Understand the physical meaning of matrix multiplication.
(2) Understand when a matrix multiplication makes sense and when it does not.
(3) Matrix multiplication is not commutative.

A digital image of size $M \times N$ can be represented using a matrix of size $M \times N$ where each cell of the matrix contains a value between 2^0 and 2^k that gives different shades of grey and k is the number of bits a computer affords. For example, an 8-bit ($k = 8$) machine has 256 (0–255) shades of grey where a value of 0 means the pixel is completely dark while a value of 255 means the pixel is completely white. Figure 1 shows how a simple black-and-white image is represented by a corresponding matrix of the respective size.

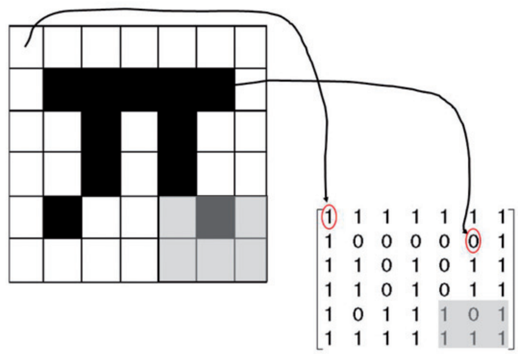

Figure 1. An illustration how an 8-bit image is represented by a matrix. In this example, black pixels are represented numerically by 0 while the white pixels are represented by 1.

Discussion:

(1) Discuss with students various ways to store information. In particular, get students to see that matrices can be used to represent digital images.

- *Question*: What are some ways to conveniently represent a collection of similar objects so we can recall the content easily?
- *Question*: How is a black-and-white digital image represented mathematically?
- *Question*: How is a colour image represented mathematically? (A colour image can be represented using a mixture of three matrices, one for red, one for green and one for blue.)

Let X be the 499×387 matrix that represents the image shown in Figure 2. In general, we denote the content that occupies mth column and nth row of the matrix X by $x_{m,n}$. For simplicity, we drop the comma between the integers m and n if they are both less than 10. For example, the fourth column and third row of the matrix X is given by the value x_{43} or $x_{4,3}$.

There are multiple ways to define matrix multiplication, both geometrically and algebraically. Geometrically, the action of AB, where A is of size $m \times n$ and B is of size $n \times p$, can be realized as performing linear transformation on column vectors of B, i.e. $AB = A[\mathbf{b}_1\ \mathbf{b}_2\ \cdots\ \mathbf{b}_p] = [A\mathbf{b}_1\ A\mathbf{b}_2\ \cdots\ A\mathbf{b}_p]$. Understanding the action of AB amounts to understanding the action of Ax, where x is a column vector. This concept may have or have not already been introduced to students in a previous lesson. In either case, it might be beneficial to remind students that the entries in $x = [x_1, x_2, \ldots, x_n]$ serve as weights in the linear combination of

$$Ax = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n.$$

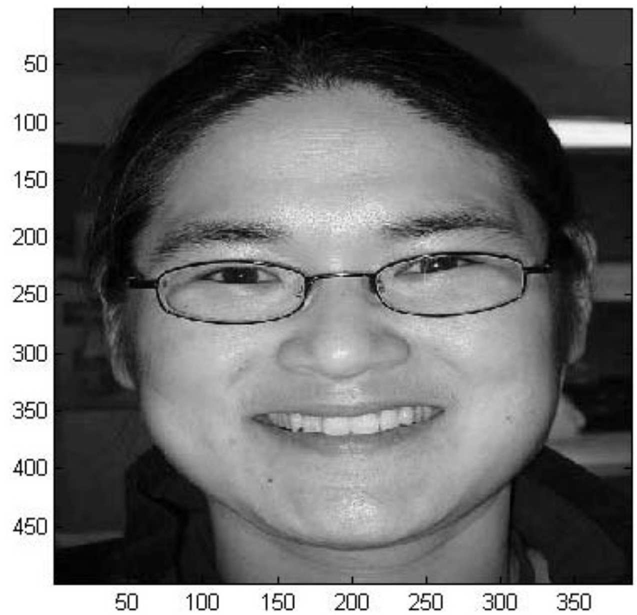

Figure 2. An image of size 499 × 387 that is represented by the matrix X.

This discussion should lend itself very well to the *conformability* of matrix multiplication, i.e. when it makes sense to multiply two matrices and the size of the resulting matrix.

On the other hand, using the purely algebraic *row–column* rule, one obtains the (i,j)th entry of AB via the expression $\sum_{k=1}^{n} a_{ik}b_{kj} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$. With the *column–row* rule,

$$AB = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix} \begin{bmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \\ \vdots \\ \mathbf{b}_n \end{bmatrix} = \sum_{i=1}^{n} \mathbf{a}_i \mathbf{b}_i,$$

where each $\mathbf{a}_i\mathbf{b}_i$ is an outer product of size $m \times p$.

Discussion:

(1) Discuss the numerical representation of matrices, i.e. $X = (x_{i,j})$, where $1 \le i \le M$ and $1 \le j \le N$, using the language of images if needed.

- *Question*: Give ways to represent the entries in X so that we can retrieve information easily.
- *Question*: What do you think it means to *multiply* two matrices? Elaborate both geometrically and physically using any definition of matrix multiplication.
- *Question*: Does it matter which order I multiply the two matrices? i.e. is $XY = YX$ in general? Why or why not?
- *Question*: Is matrix multiplication always possible for *any* size of matrices? Why or why not?

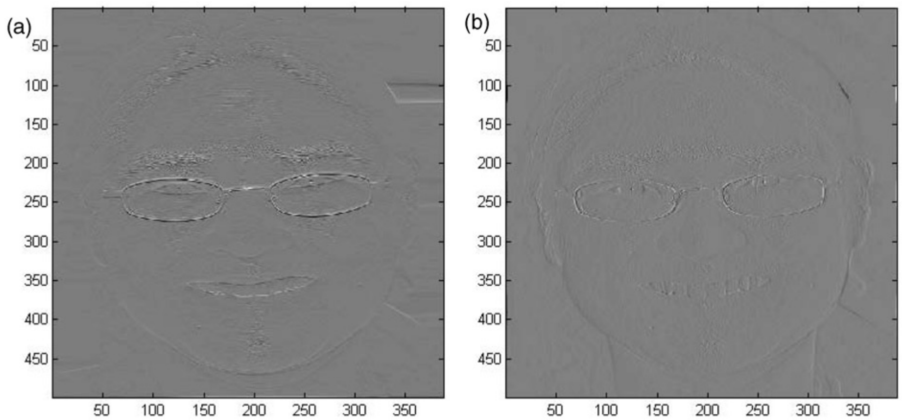

Figure 3. (a) Horizontal edges of Figure 2, accomplished by the matrix multiplication HX. (b) Vertical edges of Figure 2, accomplished by the matrix multiplication XV.

- *Question*: What does the matrix H have to look like so that

$$HX = \begin{bmatrix} x_{11} - x_{21} & x_{12} - x_{22} & \cdots & x_{1,387} - x_{2,387} \\ x_{21} - x_{31} & x_{22} - x_{32} & \cdots & x_{2,387} - x_{3,387} \\ \vdots & \vdots & \ddots & \vdots \\ x_{499,1} - x_{11} & x_{499,2} - x_{12} & \cdots & x_{499,387} - x_{499,387} \end{bmatrix}?$$

The matrix H such that

$$H = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & -1 \\ -1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 \end{bmatrix}$$

acts like a horizontal edge detector by taking differences between consecutive rows. This is because in areas where the pixels in consecutive rows do not change much in value, the components of HX are pretty close to zero (black). Conversely, whenever there is a boundary (along a row), the values of HX will be large (white), and the larger they are, the whiter the resulting pixel. So multiplying by H on the left might be thought of as a very naive method for detecting boundaries along the rows in X. See the result of HX in Figure 3(a).

Discussion:

(1) Check for student understanding on the concepts discussed so far.

- *Question*: What is the size of H?

- *Question*: Is XH possible? Why or why not?

(2) Check to see if students really master the concepts by asking them to find patterns on their own.

- *Question*: How do we compute differences along the consecutive columns of X?
- *Question*: Is it possible to find a matrix V such that

$$VX = \begin{bmatrix} x_{11} - x_{12} & x_{12} - x_{13} & \cdots & x_{1,387} - x_{11} \\ x_{21} - x_{22} & x_{22} - x_{23} & \cdots & x_{2,387} - x_{21} \\ \vdots & \vdots & \ddots & \vdots \\ x_{499,1} - x_{499,2} & x_{499,2} - x_{499,3} & \cdots & x_{499,387} - x_{499,1} \end{bmatrix}?$$

Why or why not? If so, find it. What is the size of V? If not, how do you fix it?

For the similar reasons, right multiplication by

$$V = \begin{bmatrix} 1 & 0 & \cdots & 0 & -1 \\ -1 & 1 & \cdots & 0 & 0 \\ 0 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & -1 & 1 \end{bmatrix}$$

can be thought of as a method for extracting boundaries along the columns of X, hence a vertical edge detector. Precisely,

$$XV = \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1,387} \\ x_{21} & x_{22} & \cdots & x_{2,387} \\ \vdots & \vdots & \ddots & \vdots \\ x_{499,1} & x_{499,2} & \cdots & x_{499,387} \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & \cdots & 0 & -1 \\ -1 & 1 & \cdots & 0 & 0 \\ 0 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} x_{11} - x_{12} & x_{12} - x_{13} & \cdots & x_{1,387} - x_{11} \\ x_{21} - x_{22} & x_{22} - x_{23} & \cdots & x_{2,387} - x_{21} \\ \vdots & \vdots & \ddots & \vdots \\ x_{499,1} - x_{499,2} & x_{499,2} - x_{499,3} & \cdots & x_{499,387} - x_{499,1} \end{bmatrix}.$$

The vertical edges extracted this way is shown in Figure 3(b).

Discussion:

(1) Provide other food for thoughts.

- *Question*: What is the difference between inner product and outer product?
- *Question*: Can you come up with other types of edge detectors via matrix multiplications?
- *Question*: What else can matrix multiplication mean? (For example, matrix multiplication can be thought as a form of linear transformation such as rotation and scaling.)
- *Question*: What other operations can you do to a collection of matrices/images? What result do they give you? Use the language of images if needed. (For example, multiplying -1 to every pixel turns the *negatives* into *positives*, multiplying a positive scalar to a image matrix is equivalent to turning the light intensity up, take the average of many face images produce an average human face.)

5. Orthogonal projection lesson

In this example lesson, we connect the geometric meaning of the orthogonal projection of a vector onto a subspace with an image processing algorithm called the *novelty filter*. The lesson is inspired by an example presented in [12].

Objectives:

(1) Be familiar with the geometrical interpretation of inner products.
(2) Understand the mathematical meaning and practical uses of the equation $\mathbf{y} = \hat{\mathbf{y}} + \mathbf{z}$ for high-dimensional vectors.

Students first see the concept of orthogonal projection formally in a multivariable calculus class when they are asked to decompose a (column) vector $\mathbf{y}$ into two vectors $\hat{\mathbf{y}}$ and $\mathbf{z}$ such that $\hat{\mathbf{y}}$ is in the direction of a given vector $\mathbf{u}$ and $\mathbf{z}$ is perpendicular to $\mathbf{u}$. As illustrated in Figure 4(a),

$$\mathbf{y} = \hat{\mathbf{y}} + \mathbf{z} \quad \text{and} \quad \hat{\mathbf{y}} = Proj_{\mathbf{u}}\mathbf{y} = \frac{\mathbf{y}^T\mathbf{u}}{\mathbf{u}^T\mathbf{u}}\mathbf{u}.$$

In the high-dimensional analogue, the orthogonal projection of a vector $\mathbf{y}$ onto a p-dimensional subspace, W, that is spanned by a collection of p orthogonal basis vectors $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_p\}$ is given by

$$\hat{\mathbf{y}} = \frac{\mathbf{y}^T\mathbf{u}_1}{\mathbf{u}_1^T\mathbf{u}_1}\mathbf{u}_1 + \frac{\mathbf{y}^T\mathbf{u}_2}{\mathbf{u}_2^T\mathbf{u}_2}\mathbf{u}_2 + \cdots + \frac{\mathbf{y}^T\mathbf{u}_p}{\mathbf{u}_p^T\mathbf{u}_p}\mathbf{u}_p = \hat{\mathbf{y}}_1 + \hat{\mathbf{y}}_2 + \cdots + \hat{\mathbf{y}}_p.$$

If we imagine that $W = W_1 \oplus W_2 \oplus \cdots \oplus W_p$, where each $W_i = \text{span}\{\mathbf{u}_i\}$ is the 1-D subspace spanned by $\mathbf{u}_i$, then the expression $\frac{\mathbf{y}^T\mathbf{u}_i}{\mathbf{u}_i^T\mathbf{u}_i}\mathbf{u}_i = \hat{\mathbf{y}}_i$ is the orthogonal projection of $\mathbf{y}$ onto each W_i, for $i = 1, 2, \ldots, p$. Figure 4(b) gives a visual presentation for $p = 2$. Notice that this formulation is only possible under the assumption that $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_p\}$ forms an *orthogonal* basis for W. The formula simplifies further if the basis is *orthonormal*, i.e. if $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_p\}$ is an orthonormal basis for W, then

$$\hat{\mathbf{y}} = (\mathbf{y}^T\mathbf{u}_1)\mathbf{u}_1 + (\mathbf{y}^T\mathbf{u}_2)\mathbf{u}_2 + \cdots + (\mathbf{y}^T\mathbf{u}_p)\mathbf{u}_p.$$

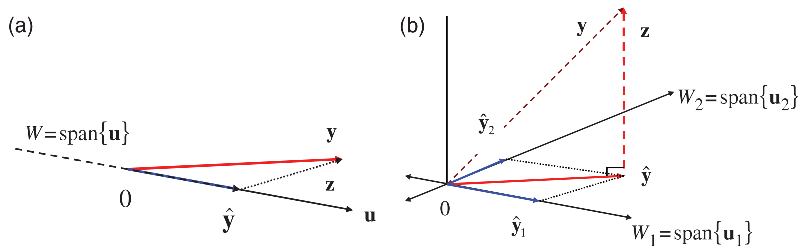

Figure 4. Geometric illustrations of $\mathbf{y} = \hat{\mathbf{y}} + \mathbf{z}$. (a) $\hat{\mathbf{y}}$ is the orthogonal projection of $\mathbf{y}$ onto the 1-D subspace spanned by $\mathbf{u}$. (b) $\hat{\mathbf{y}}$ is the orthogonal projection of $\mathbf{y}$ onto the 2-D subspace spanned by $\mathbf{u}_1$ and $\mathbf{u}_2$. Notice that $\hat{\mathbf{y}} = \hat{\mathbf{y}}_1 + \hat{\mathbf{y}}_2$, where $\hat{\mathbf{y}}_i$ is the orthogonal projection of $\mathbf{y}$ onto the subspace spanned by $\mathbf{u}_i$ for $i = 1, 2$.

This fact can be represented via a matrix equation

$$\hat{\mathbf{y}} = QQ^T\mathbf{y},$$

where $Q = [\mathbf{u}_1 \,|\, \mathbf{u}_2 \,|\, \cdots \,|\, \mathbf{u}_p]$ and QQ^T serves as a projection matrix that takes $\mathbf{y}$ onto W.

In a way, the vector $\hat{\mathbf{y}}$ is the part of $\mathbf{y}$ that can be represented by vectors in W and the vector $\mathbf{z}$, which lives in the orthogonal complement, $W^\perp$, of W is the part of $\mathbf{y}$ that could not be represented by vectors in W. Therefore, $\mathbf{z}$ measures how different the vector $\mathbf{y}$ is from the subspace W. For this reason, we call $\mathbf{z}$ the *novelty* (or *residual*) of $\mathbf{y}$ when compared to vectors from W.

Discussion:

(1) Get students to see the geometric meaning of inner product in 2-D as well as in higher dimensions.

- *Question*: What is the orthogonal projection of a vector $\mathbf{y} \in \mathbb{R}^2$ onto another vector $\mathbf{u} \in \mathbb{R}^2$? (Recall definitions of dot product from Calculus if needed.)
- *Question*: What is the orthogonal projection of a vector $\mathbf{y} \in \mathbb{R}^3$ onto a subspace W spanned by linearly independent vectors $\mathbf{u}_1$ and $\mathbf{u}_2$ in $\mathbb{R}^3$? Draw a picture to illustrate your ideas.
- *Question*: Generalize your idea to vectors in higher dimensions and find the orthogonal projection of $\mathbf{y} \in \mathbb{R}^n$ onto a p-dimensional subspace W that is spanned by $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_p\}$.
- *Question*: Why is the fact that $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_p\}$ forms an orthogonal basis necessary in the expression of $\hat{\mathbf{y}}$? (Consider writing $\mathbf{y}$ as a linear combination of $\mathbf{u}_i$'s, i.e. $\mathbf{y} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_p\mathbf{u}_p$. What happens if the $\mathbf{u}_i$'s do not form an orthogonal basis?)

Now imagine that we are given three (rasterized) 5×4 images, as shown in Figure 5. Assume that the *black* square entries have numerical value 1 and the *blank* entries have numerical value 0. These images can be realized as points in $\mathbb{R}^{20}$ after column concatenation and serve as the basis for W, i.e. $W = \text{span}\{v_1, v_2, v_3\}$. Moreover, let $\mathbf{y}$ be a pattern that does not live in W, as shown in Figure 6. That is, there is no way that we can come up with $\mathbf{y}$ by taking linear combinations of $\mathbf{v}_i$'s.

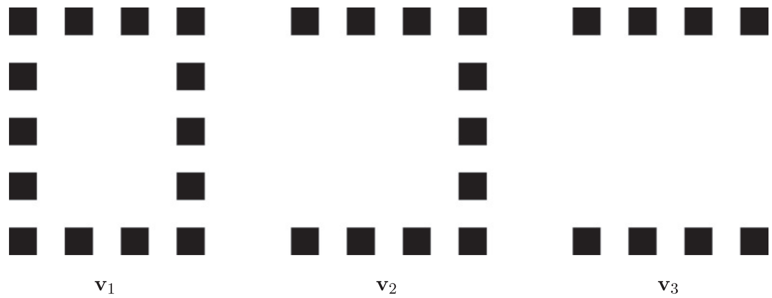

Figure 5. Basis elements for $W = \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ that are used as a gallery/training set.

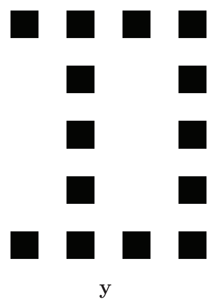

Figure 6. A given pattern that is outside of W, used as a probe.

Thus, to figure out how different (or novel) $\mathbf{y}$ is from the space generated by $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, the first thing we need to do is to find an **orthogonal basis** for $W = \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, call it $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$. Then the amount of *novelty* is given by the residual vector $\mathbf{y} - \hat{\mathbf{y}}$, where $\hat{\mathbf{y}} = \sum_{i=1}^{3} \hat{\mathbf{y}}_i$. This can be accomplished by finding the projection of $\mathbf{y}$ onto each W_i, denoted by $\hat{\mathbf{y}}_i$, followed by a sum. The result of the three respective orthogonal projections is shown in Figure 7. Each $\hat{\mathbf{y}}_i$ gives the part of $\mathbf{y}$ that can be represented using vectors in W_i. For example, W_3 is sort of redundant since elements in W_2 can already describe the first and last row of squares in $\mathbf{v}_3$, therefore $\hat{\mathbf{y}}_3$ is close to a zero vector. For plotting purposes, the imagesc command in MATLAB automatically re-scales the image from its dynamic range to the entire interval $[0, 255]$ in order to make grey values that are close to zero visible. This explains the greyish effect in Figure 7.

We can then write $\mathbf{y}$ as a sum of orthogonal vectors, one in W and one in the orthogonal complement of W, $W^{\perp}$, i.e. $\hat{\mathbf{y}} = \hat{\mathbf{y}} + \mathbf{z}$, where $\hat{\mathbf{y}} \in W$ and $\mathbf{z} \in W^{\perp}$. See Figure 8 for this graphical result. Notice that the three vertical squares in the second column of $\mathbf{y}$ appears to be different (novel) from the space W, hence cannot be generated from any linear combination of vectors in W.

Discussion:

(1) Get students to see the practical use of high-dimensional orthogonal projections.

- *Question*: Why can we realize these 5×4 images as points in $\mathbb{R}^{20}$?

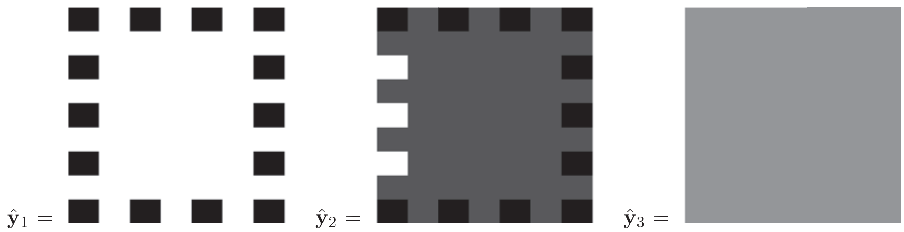

Figure 7. Orthogonal projections of $\mathbf{y}$ onto the subspace $W = \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ expressed separately in each subspace spanned by $\mathbf{u}_i$, $i = 1, 2, 3$.

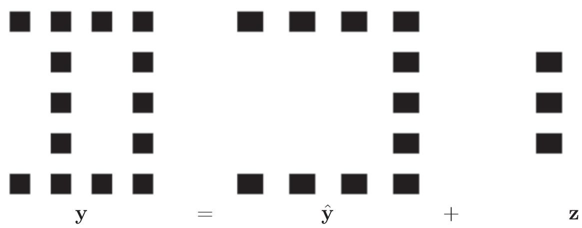

Figure 8. $\mathbf{y}$ Written as a sum of two orthogonal vectors $\hat{\mathbf{y}}$ and $\mathbf{z}$, where $\hat{\mathbf{y}} \in W$ and $\mathbf{z} \in W^{\perp}$.

- *Question*: How is $\mathbf{y}$ different from the space spanned by $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$? That is, what is the novelty (or residual) when you orthogonally project $\mathbf{y}$ onto $W = \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$? (We are essentially looking for the part of $\mathbf{y}$ that can not be represented by the $\mathbf{v}_i$'s, which are the three squares down the second column.)
- *Question*: What does each $\hat{\mathbf{y}}_i$ mean in this context? What do $\hat{\mathbf{y}}$ and $\mathbf{z}$ mean in this context?
- *Question*: Can you think of a situation where performing orthogonal projection can be useful? (For example, anomaly and motion detection.)

6. MATLAB implementation

Upon loading the MATLAB program, save the following to a blank script file. Make sure your image file can be found in the current directory or direct the code to the appropriate path as shown in this sample. The codes provided here are not optimized numerically and are meant to only provide convenience for the readers. The purpose of having codes available to instructors is to allow immediate modifications depending on instructional needs. The accessibility and portability of programs is a major deciding factor in lesson adoption. Comments in the code appear after %.

6.1. *Matrix multiplication lesson*

```
clear
X = imread('../data/myImage.jpg');
%% comment out the line above and uncomment the line below
```

```
%% if the image is in the current directory
% X = imread('myImage.jpg');
cd ../ %% up one level to working directory

Y = rgb2gray(X); %% convert myImage.jpg to monochrome
pattern = double(Y); %% convert myImage.jpg to double precision
[m,n] = size(pattern); %% reads in the image size

%% construct horizonal edge detector
H = eye(m,m);
for i = 1:m-1
  H(i,i+1) = -1;
end
H(m,1) = -1;
RowEdge = H*pattern; %% perform edge detection

%% construct vertical edge detector
H = eye(n,n);
for i = 1:n-1
  H(i+1,i) = -1;
end
H(n,1) = -1;
ColEdge = pattern*H; %% perform edge detection

%% plot the results in separate windows
figure, imagesc(pattern), colourmap(gray), axis square
title('Original Image')
figure, imagesc(RowEdge), colourmap(gray), axis square
title('Horizontal Edges')
figure, imagesc(ColEdge), colourmap(gray), axis square
title('Vertical Edges')
```


6.2. *Orthogonal projection lesson*

```
%% black square = 1, blank square = 0
%% create gallery points
v1 = [1,1,1,1,1,1,0,0,0,1,1,0,0,0,1,1,1,1,1,1]';
v2 = [1,0,0,0,1,1,0,0,0,1,1,0,0,0,1,1,1,1,1,1]';
v3 = [1,0,0,0,1,1,0,0,0,1,1,0,0,0,1,1,0,0,0,1]';

%% probe
y = [1,0,0,0,1,1,1,1,1,1,1,0,0,0,1,1,1,1,1,1]';

%% obtain orthogonal basis for the training subspace
V = [v1 v2 v3];
[Q,R] = qr(V,0);
u1 = Q(:,1);
u2 = Q(:,2);
u3 = Q(:,3);

%% orthogonal projection onto each direction
y1_hat = ((y'*u1)/(u1'*u1)).*u1;
```

```
y2_hat = ((y'*u2)/(u2'*u2)).*u2;
y3_hat = ((y'*u3)/(u3'*u3)).*u3;

y_hat = y1_hat + y2_hat + y3_hat;
%% alternatively, y_hat = Q*Q'*y;
z = y - y_hat; %% residual/novelty

%% graph the results:
I = ones(5,4);
temp1 = reshape(y1_hat,5,4); temp1 = I - temp1;
figure, imagesc(temp1), colourmap(gray), axis off

temp2 = reshape(y2_hat,5,4); temp2 = I - temp2;
figure, imagesc(temp2), colourmap(gray), axis off

temp3 = reshape(y3_hat,5,4); temp3 = I - temp3;
figure, imagesc(temp3), colourmap(gray), axis off

y_hat = reshape(y_hat,5,4); y_hat = I - y_hat;
figure, imagesc(y_hat), colourmap(gray), axis off

z = reshape(z,5,4); z = I - z;
figure, imagesc(z), colourmap(gray), axis off
```


7. Summary and discussions

Teaching and learning should be thought as a two-way information exchange. Skilfully-posed questions allow instructors to hear how students process new information and pinpoint the source of errors while it also gives students a chance to be actively involved in the problem-solving process. The use of applications at the *beginning* of a lesson makes the concepts relevant and provides students with a concrete grasp of abstraction. Designing lessons that incorporate both of these two ideas may not be realistic on a daily basis; however, the practice of thinking in this framework should be emphasized and carried over to all areas of mathematics.

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Deal or No Deal: using games to improve student learning, retention and decision-making†

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Student understanding and retention can be enhanced and improved by providing alternative learning activities and environments. Education theory recognizes the value of incorporating alternative activities (games, exercises and simulations) to stimulate student interest in the educational environment, enhance transfer of knowledge and improve learned retention with meaningful repetition. In this case study, we investigate using an online version of the television game show, 'Deal or No Deal', to enhance student understanding and retention by playing the game to learn expected value in an introductory statistics course, and to foster development of critical thinking skills necessary to succeed in the modern business environment. Enhancing the thinking process of problem solving using repetitive games should also improve a student's ability to follow non-mathematical problem-solving processes, which should improve the overall ability to process information and make logical decisions. Learning and retention are measured to evaluate the success of the students' performance.

Keywords: statistics; expected value; experiential learning

1. Introduction

For a number of years, educators have recognized that experiential techniques and alternative learning environments are useful in helping students better understand and retain information. For example, according to the proponents of Activity Theory [1–3], learning is conceptualized not just as a function of a game itself, but

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