Interpretations of the Common Core State Standards (**CCSS**) in Mathematical Modeling

Jen-Mei Chang, Ph.D.

Department of Mathematics and Statistics
College of Natural Sciences and Mathematics
California State University, Long Beach
jen-mei.chang@csulb.edu

EQALS Institute, CSU Long Beach
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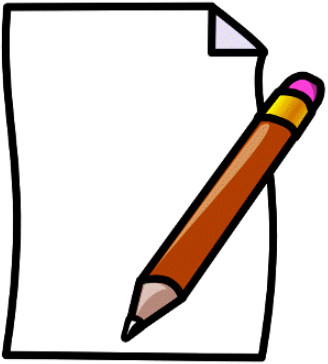
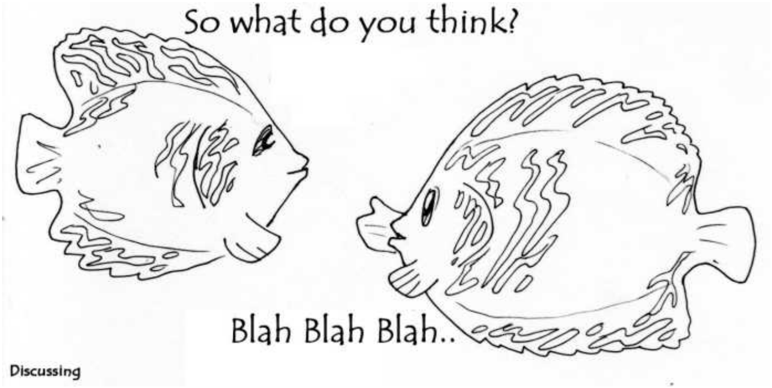
Outline

Think & Write



Discuss



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M.M. by **CCSS**

Modeling is the process of choosing and using appropriate mathematics and statistics to analyze empirical situations, to understand them better, and to improve decisions. – **CCSS**

Example

- What is π?
- Since π is transcendental (not a root of any polynomial with rational coefficients), there is not an exact expression to represent it.
- Thus, a choice needs to be made in ways to model it.
- Question is, how should we model it and why do we model it a certain way?

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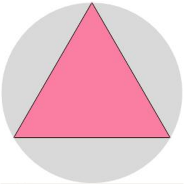
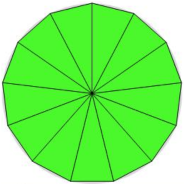
Geometer's Take For π

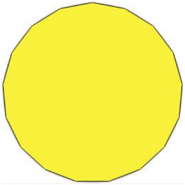
- Assuming the world is filled with flat objects (Euclidean geometry), π is defined as the ratio of a circle's circumference to its diameter.
- (inspired by Archimedes, 250 B.C.) π can be modeled by the areas of inscribed regular n-gons in a unit circle.



$n = 3, \pi \approx 1.2993$



$n = 13, \pi \approx 3.0207$



$n = 17, \pi \approx 3.0706$

Algebraist's Take For π

- π can be modeled by continued fractions. e.g.,

$$\pi = 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + \cfrac{1}{292 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \ddots}}}}}}}$$

Analyst's Take For π

- π can be modeled by infinite series.
- For example, since

$$\tan^{-1} z = z - \frac{z^3}{3} + \frac{z^5}{5} - \frac{z^7}{7} + - \cdots,$$

thus with $z = 1$, we have

$$\frac{\pi}{4} = \tan^{-1} 1 = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + - \cdots.$$

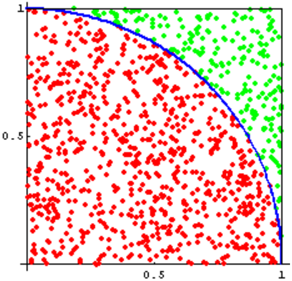
This then gives us an estimate of π to any precision (though slow convergence: accurate to 5 decimal digits after 500,000 terms):

$$\pi = 4 - \frac{4}{3} + \frac{4}{5} - \frac{4}{7} + \frac{4}{9} - + \cdots$$

Statistician's Take For π

- π can be modeled by the Monte Carlo method.
- Draw a unit circle inscribed in a square (of length 2) and randomly place dots in the square. The ratio, ρ, of number of dots inside the circle to the total number of dots is $\approx \frac{\pi}{4}$, since

$$\frac{\text{area of circle}}{\text{area of square}} = \frac{\pi \cdot 1^2}{2^2} = \frac{\pi}{4}$$



- In a typical simulation of sample size 1000, there are 787 points satisfying $x^2 + y^2 \leq 1$.
- Using this data, $\rho = \frac{787}{1000}$. Hence $\pi \approx 4 \cdot \rho = 3.148$.

Observations

What message did you get from the π example?

- Depending on one's training and interests, modeling of a task can be done with very different approaches.
- It is essential that we gain a deep insight in all areas of mathematics.

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Mathematics as Structures

The same mathematical or statistical structure can sometimes model seemingly different situations. – **CCSS**

Example

- A **matrix** is a rectangular array of numbers, symbols, or expressions, arranged in rows and columns.
- How can matrices be used to model real-world situations?

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Matrix Structure in Linear Algebra

- Consider the following system of 4 equations in 4 unknowns:

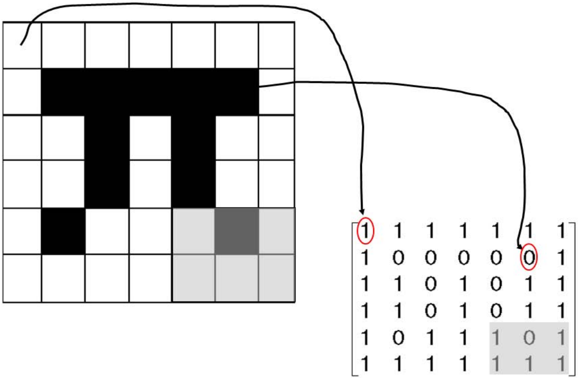
$$\begin{aligned} x_1 + 3x_2 - 4x_3 + 9x_4 &= 5 \\ -x_1 + 2x_2 + 3x_3 - 14x_4 &= 8 \\ 3x_1 - 6x_2 + 5x_3 + 11x_4 &= 7 \\ 5x_1 - 9x_2 - 2x_3 + 17x_4 &= 23 \end{aligned}$$

- The system can be expressed as $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} 1 & 3 & -4 & 9 \\ -1 & 2 & 3 & -14 \\ 3 & -6 & 5 & 11 \\ 5 & -9 & -2 & 17 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \text{ and } \quad \mathbf{b} = \begin{bmatrix} 5 \\ 8 \\ 7 \\ 23 \end{bmatrix}$$

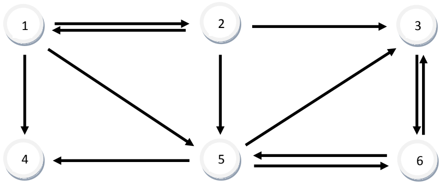
Matrix Structure in Digital Imaging

An *r*-by-*c* gray scale (black and white) digital image corresponds to an *r*-by-*c* matrix where each entry in the image matrix enumerates one of the overall possible gray levels of the corresponding pixel. e.g., matrix representation of a binary image:



Matrix Structure in Graph Theory

- **Google** uses a link graph to study the connection between each web page and solves the eigen-problem $Qr = 1r$ to find out the rank of each page for a particular query.
- e.g., if the following six web pages are linked as such



- Then the corresponding linkage matrix is

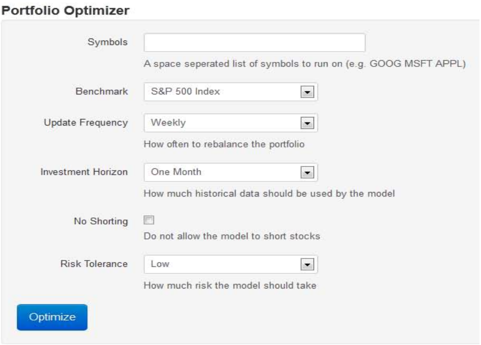
$$Q = \begin{bmatrix} 0 & \frac{1}{3} & 0 & 0 & 0 & 0 \\ \frac{1}{3} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 & \frac{1}{3} & \frac{1}{2} \\ \frac{1}{3} & 0 & 0 & 0 & \frac{1}{3} & 0 \\ \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 0 & \frac{1}{3} & 0 \end{bmatrix}.$$

Varying Parameters

When making mathematical models, technology is valuable for varying assumptions, exploring consequences, and comparing predictions with data.
– **CCSS**

Example

Suppose you are curious what kind of investment portfolio will pay off big time in the next 5 years. How do you make an intelligent decision on what combination will work the best?



Compare Model to Real Data

When making mathematical models, technology is valuable for varying assumptions, exploring consequences, and comparing predictions with data.
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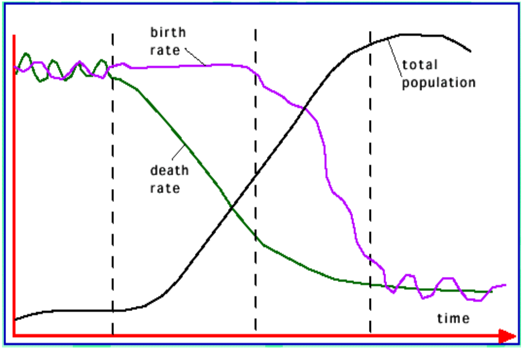
Example

Does world population increase over time (indefinitely)? What is a good model to make accurate prediction of world population for year 2050?



Observations

What message did you get from the matrix structure example and the numerical examples?

- We need to develop an appreciation for the theoretical as well as the computational/numerical aspect of mathematics.
- As educators, we need to design curricula that allow students to develop the necessary skill set to be fluent in both areas.

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A Simple M.M. Problem

A model can be very simple, such as writing total cost as a product of unit price and number bought. – **CCSS**

Example

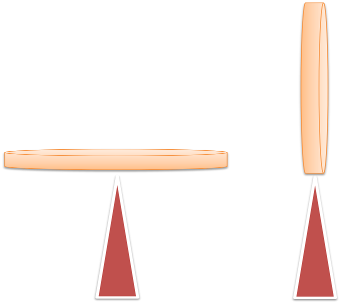
Cost $= n \cdot P$, where n denotes the number of items purchased and P is the price per item in dollars.

Decision, Decision, Decision

A model can be very simple, such as using a geometric shape to describe a physical object like a coin. Even such simple models involve making choices. It is up to us whether to model a coin as a three-dimensional cylinder, or whether a two-dimensional disk works well enough for our purposes. – **CCSS**

Example

How should we model the coin if we want to find the locations to balance the coin?



Modeling Arises From Necessity

Other situations – modeling a delivery route, a production schedule, or a comparison of loan amortization – need more elaborate models that use other tools from the mathematical sciences. – **CCSS**

Example

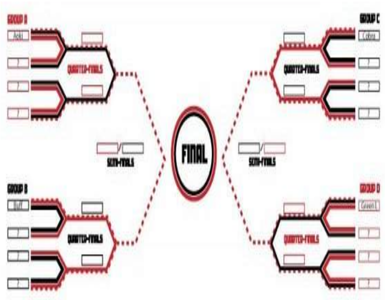
A traveling salesman problem (TSP): Given a list of cities and their pairwise distances, the task is to find the shortest possible route that visits each city exactly once and returns to the origin city.

Real-World Modeling Problems

Real-world situations are not organized and labeled for analysis; formulating tractable models, representing such models, and analyzing them is appropriately a creative process. Like every other process, this depends on acquired expertise as well as creativity. – **CCSS**

Example

Planning a table tennis tournament for 7 players at a club with 4 tables, where each player plays against each other player.

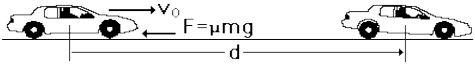


Decision, Decision, Decision

In situations like these, the models devised depend on a number of factors: How precise an answer do we want or need? What aspects of the situation do we most need to understand, control or optimize? What resources of time and tools do we have? – **CCSS**

Example

Analyzing stopping distance for a car.



Why Model a Certain Way

A **less-sophisticated** model

Assume the average car decelerates at about 15 feet per second (fps) per second.

{ For public awareness }

A **more-sophisticated** model

$\text{Work}_{\text{friction}} = -\mu mgd = -\frac{1}{2}mv_0^2$, where $\mu =$ effective coefficient of friction between the tires and the road which varies from car to car. Then, $d = \frac{v_0^2}{2\mu g}$.

{ For design and performance test }

The stopping distance for a car driving at 60mph ($\approx$ 88fps):

- In the **less-sophisticated** model, with 1 second delay in reaction, $88 \to 73 \to 58 \to 43 \to 28 \to 13 \to 0$, the total distance is roughly 303.05 feet (which is a little longer than a football field).
- In the **more-sophisticated** model, with a μ value of 0.75 for an average car, the total distance is roughly $\frac{88^2}{2 \cdot 0.75 \cdot 32} = 161.37$ feet.

Observations

What message did you get from the given modeling examples?

- The need to model situations mathematically is a consequence to make our lives easier.
- A lot of decisions need to be made depending on the purpose of the modeling task. Consequently, mathematical modeling is an open-ended process.

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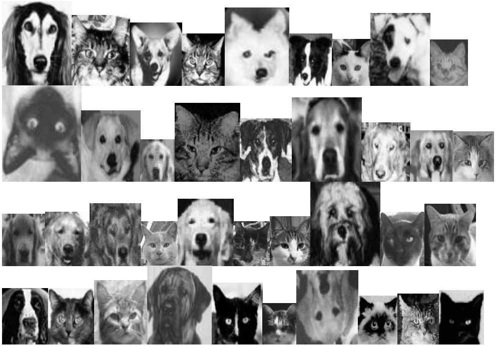
CCSS Basic Modeling Cycle Exercise

Objective: Create a model to differentiate images of cats from images of dogs.

Problem Solving Strategy

- Draw diagram and visualize
- Determine underlying principles
- Determine methods of solving
- Execute
- Be skeptical



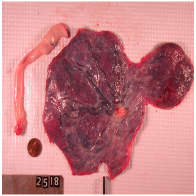
Basic Modeling Cycle – **CCSS**

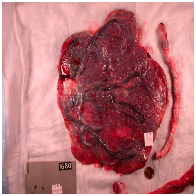
1. *Identify variables in the situation and select those that represent essential features*
2. *Formulate a model by creating and selecting geometric, graphical, tabular, algebraic, or statistical representations that describe relationships between the variables*
3. *Analyze and perform operations on these relationships to draw conclusions*
4. *Interpret the results of the mathematics in terms of the original situation*
5. *Validate the conclusions by comparing them with the situation, and then either improve the model or, if it is acceptable*
6. *Report on the conclusions and the reasoning behind them*

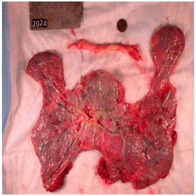
CCSS Basic Modeling Cycle Exercise

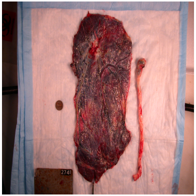
Objective: Design a model that doctors can use to inform parents whether their newborn is healthy based on the image of mother's placenta after birth.

1. Features, variables?
2. Relationship between variables?
3. Conjectures, hypothesis?
4. How to verify your hypothesis?
5. How do you know your method is correct? When do you stop?









Summary

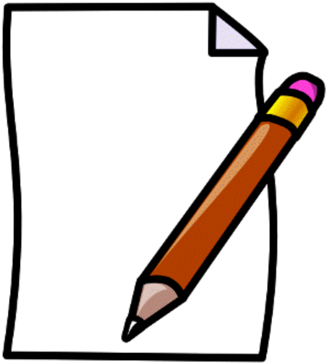
- Modeling starts with questions; and questions arise from curiosity. This is what we need to emphasize when teaching mathematics – problem-solving as well as content knowledge.
- Different people model the same problem differently. Why?
- There are numerous tools available to assist us in the modeling process. For example,
  - graphing utilities (such as TI-calculators and MS Excel);
  - data analysis software (such as MS Excel spreadsheets and SAS);
  - dynamic geometry software (such as GeoGebra and Dynamic Geometry);
  - computer algebra systems (such as MATLAB, Maple, Mathematica).

Summary Continued

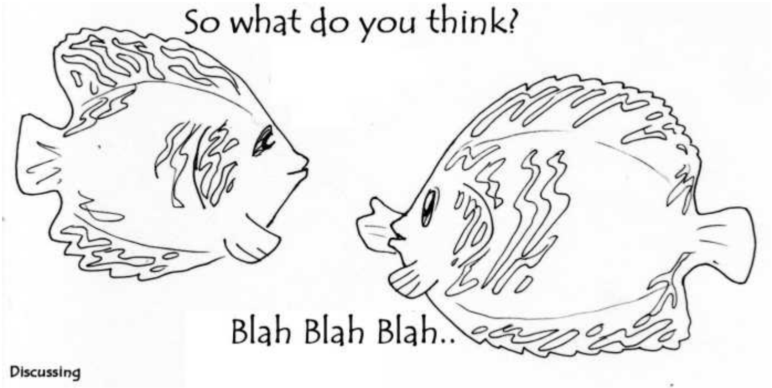
- Hence, numerical aspect of mathematical knowledge is just as important as the theoretical one.
- There is not a single way to learn how to model real-life phenomena; rather, it is the collective knowledge that we acquire during our mathematical training that helps us interpret the problems.
- It is particularly important that we understand very well what mathematical functions and expressions do before we can accurately model situations with them.

Reflections

So, how should we teach students to model real-world situations mathematically?



Discuss



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Modeling is best interpreted not as a collection of isolated topics but rather in relation to other standards.
— **CCSS**