CSULB Math Day Noon Entertainment: An Academic Journey of “Tangent”

JEN-MEI CHANG

Department of Mathematics and Statistics
California State University, Long Beach
jchang9@csulb.edu

π Day, 2009
Outline

1. High School
   - Geometry Class
   - Algebra Class

2. College
   - Calculus I
   - Calculus II
   - Calculus III

3. Graduate School
   - Pure Math
   - Applied Math

4. Real Life

5. Conclusions
Geometrically

What is a Tangent line geometrically?
Algebraically

What is the equation of a **Tangent** line and why?

\[ y - 1 = \frac{1}{2}(x - 1) \quad \text{or} \quad y = \frac{1}{2}x + \frac{1}{2} \]
Algebraically

What is the equation of a Tangent line and why?

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Average R.O.C. and Secant

1. Average R.O.C between $P$ and $Q$ is given by $\frac{f(1 + h) - f(1)}{(1 + h) - 1}$.

2. But what does $\frac{f(1 + h) - f(1)}{(1 + h) - 1}$ also represent? Slope of the secant lines.
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**Journey of “Tangent”**

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\] also represent? Slope of the secant lines.
Instantaneous R.O.C. and Tangent

1. **Instantaneous R.O.C at** \( P \) **is given by** \( \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} \).

2. So what does \( \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} \) also represent? **Slope of the tangent line**.
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First Degree Approximation to Curves

**Question:** What is $\sqrt{2}$?

**Geometry of Tangent Approximation:**

- Exact value = $\sqrt{2}$.
- Tangent approximation gives 1.5. But is it accurate?
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- **Exact value** = $\sqrt{2}$.
- **Tangent approximation** gives 1.5. But is it accurate?
Taylor Approximation

English mathematician Brook Taylor came up with a way to approximate functions $f(x)$ at a point $x = a$ using polynomials:

$$f(x) = \sum_{n=1}^{\infty} \frac{f^{(n)}(a)}{n!} (x - a)^n = f(a) + \frac{f'(a)}{1!} (x - a) + \frac{f''(a)}{2!} (x - a)^2 + \frac{f'''(a)}{3!} (x - a)^3 + \cdots$$

The truncation

$$f(x) \approx f(a) + f'(a)(x - a)$$

is called the Tangent approximation.
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Taylor Approximation for $\sqrt{2}$

So for $f(x) = \sqrt{x}$ with $a = 1$, we can approximate $\sqrt{2}$ using the series

$$\sqrt{x} \approx 1 + \frac{1}{2}(x - 1) - \frac{1}{8}(x - 1)^2 + \frac{1}{16}(x - 1)^3$$

$$- \frac{5}{128}(x - 1)^4 + \frac{7}{256}(x - 1)^5 + \cdots$$

Now, for $x = 2$

$$\sqrt{2} \approx 1 + \frac{1}{2} - \frac{1}{8} + \frac{1}{16} - \frac{5}{128} + \frac{7}{256} - \frac{21}{1024} + \frac{33}{2048} - \frac{429}{32768} + \cdots$$

According to Texas Instruments: “most of our graphing calculators use the common Taylor Series to calculate some functions. This is especially true for calculators with CAS, such as the TI-89 Titanium and TI-Nspire CAS.”
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Calculate $\sqrt{2}$ By Hand

If you wonder how people calculated square roots in the stone ages (time before scientific calculators, slide rules, and square root tables):

$$\sqrt{2.000000}$$
Calculate $\sqrt{2}$ By Hand

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\[
\begin{array}{c}
\sqrt{2.000000} \\
1
\end{array}
\]
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\[ \sqrt{2.000000} \]

\[ \begin{array}{c|c}
{1} & 1 \\
\hline
{1} & \\
\end{array} \]
Calculate $\sqrt{2}$ By Hand

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\[\sqrt{2.000000} \rightarrow 1.00 \]

Journey of “Tangent”
Calculate $\sqrt{2}$ By Hand

If you wonder how people calculated square roots in the stone ages (time before scientific calculators, slide rules, and square root tables):

\[
\sqrt{2.000000} = \frac{1}{1} \rightarrow \frac{1}{2} \times 2
\]

\[
\begin{array}{c|c}
1 & 1.00 \leftarrow 2 \\
1 & 2
\end{array}
\]
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\[
\begin{array}{c|c}
\sqrt{2.000000} & 1.4 \\
-1 & 24 \\
\hline
100 & 4 \\
-96 & \\
\hline
400 & 96 \text{ silently}
\end{array}
\]
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\[
\begin{array}{c}
\sqrt{2.000000} \\
1 \\
100 \\
-96 \\
400 \\
\end{array}
\begin{array}{c}
1.41 \\
24 \\
4 \\
281 \\
1 \\
\end{array}
\]
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\[
\begin{array}{c|c}
\sqrt{2.000000} & 1.41\
\hline
-1 & \\
\hline
1.00 & 2.4\
\hline
-0.96 & 4\
\hline
0.04 & 281\
\hline
-0.281 & 282\
\hline
0.00 & \\
\end{array}
\]
Calculate $\sqrt{2}$ By Hand

If you wonder how people calculated square roots in the stone ages (time before scientific calculators, slide rules, and square root tables):

\[\begin{array}{c|c|c}
2 & 000000 & 1.414 \\
1 & 00 & 2.4 \\
96 & 281 & 28.1 \\
400 & 2824 & 28.24 \\
11900 & 11296 & 28.28 \\
\hline
11296 & 2828 &
\end{array}\]
Calculate $\sqrt{2}$ By Hand

If you wonder how people calculated square roots in the stone ages (time before scientific calculators, slide rules, and square root tables):

\[
\begin{array}{c|c}
2.00000 & 1.414 \ldots \\
1 & 2.4 \\
\hline
1.00 & + 4 \\
96 & \downarrow 281 \\
\hline
400 & \downarrow 2824 \\
281 & \downarrow 2828 \\
\hline
11900 & \\
11296 & \\
\hline
2828 & \\
\end{array}
\]
The idea of **Tangent** approximations can be generalized in higher dimensions. For example, a 2-dimensional **Tangent plane** is a linear approximation to a 3-dimensional surface.

i.e., points on the surface can be approximated by points on the plane.
Tangent Plane

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i.e., points on the surface can be approximated by points on the plane.
Similarly, we can use a **Tangent space** to approximate a high-dimensional surface (e.g., a manifold).

How do we apply this idea to a realistic problem?
Tangent Space

- Similarly, we can use a **Tangent space** to approximate a high-dimensional surface (e.g., a manifold).

- How do we apply this idea to a realistic problem?
Problem Definition - Globally

Santa thought to himself, “only if these mails can go to the right place according to their zip code”.

Journey of “Tangent”
Handwritten Digit Classification

Problem. (Human) handwritten digits are sometimes very hard to recognize, even by human operators.

Importance. Accurate identification of the digits ensures a reliable delivery system.

Beneficiaries. Postal services (mail sorting), seaports (cargo registration), etc.

Even Santa Clause can benefit from an efficient digit classification algorithm.
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Problem Definition - Locally

How do we tell a bunch of 4’s from a bunch of 9’s?
Problem Definition - Locally

Or, how do we tell whether a new digit is a 4 or a 9?
Digit Manifolds

Imagine a high-D surface (red curve) where all 4’s live on and a high-D surface (blue curve) where all 9’s live on.
Tangent Spaces - Training

Create a **Tangent Space** of the 4’s at $F$ and create a **Tangent Space** of the 9’s at $N$.

Dimensions of the tangent spaces depend on the degree of variations.
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Create a **Tangent Space** of the 4’s at $F$ and create a **Tangent Space** of the 9’s at $N$.

Dimensions of the tangent spaces depend on the degree of variations.
Euclidean Distance

- Euclidean distance between each pair of 4 and 9 varies drastically.
- Calculation is time-consuming.
Euclidean Distance

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Tangent Distance

- Tangent distance captures the geometry.
- Calculation is efficient.
Tangent Distance

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Tangent Distance

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Classification

So, is it a 4 or a 9?
Classification Result

\[ \sum = 9 \]
Face Recognition
Face Recognition

\[
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 \\
1 & 0 & 1 & 1 & 1 \\
1 & 1 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 1 \\
1 & 1 & 1 & 0 & 1 \\
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
\vdots \\
1 & 1 & 1 & 1 \\
\end{bmatrix}
\]

\textbf{IMAGE → MATRIX → VECTOR}
Face Recognition
Self-reflection Time

What did you/we learn from this?

1. Mathematics is cool and mathematicians rock.
2. Mathematics is the foundation in answering most scientific questions and we can not live without it.

The bottom line is that I hope I have helped answering the age-old question,

"Why I have to learn the math I learn in high school"

— THANK YOU FOR YOUR ATTENTION —
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