Classification on the Grassmannians: Theory and Applications

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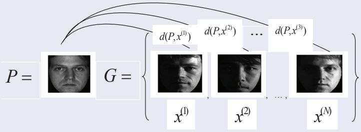
Outline

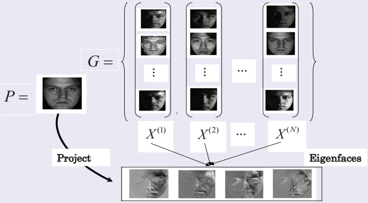
Architectures

Historically

- single-to-single

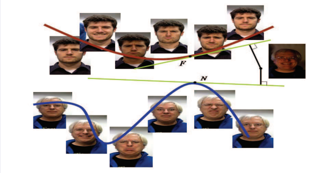


- single-to-many

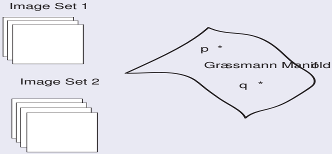


Currently

- subspace-to-subspace



- many-to-many



Some Approaches

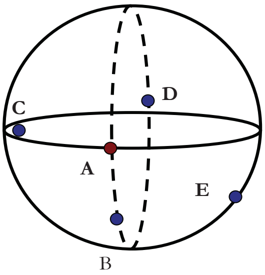
- Single-to-Single
  1. Euclidean distance of feature points.
  2. Correlation.
- Single-to-Many
  1. Subspace method [Oja, 1983].
  2. Eigenfaces (a.k.a. Principal Component Analysis, KL-transform) [Sirovich & Kirby, 1987], [Turk & Pentland, 1991].
  3. Linear/Fisher Discriminate Analysis, Fisherfaces [Belhumeur et al., 1997].
  4. Kernel PCA [Yang et al., 2000].

Some Approaches

- Many-to-Many
  1. Tangent Space and Tangent Distance - Tangent Distance [Simard et al., 2001], Joint Manifold Distance [Fitzgibbon & Zisserman, 2003], Subspace Distance [Chang, 2004].
  2. Manifold Density Divergence [Fisher et al., 2005].
  3. Canonical Correlation Analysis (CCA):
     - Mutual Subspace Method (MSM) [Yamaguchi et al., 1998],
     - Constrainted Mutual Subspace Method (CMSM) [Fukui & Yamaguchi, 2003],
     - Multiple Constrained Mutual Subspace Method (MCMSM) [Nishiyama et al., 2005],
     - Kernel CCA [Wolf & Shashua, 2003],
     - Discriminant Canonical Correlation (DCC) [Kim et al., 2006],
     - Grassmann method [Chang et al., 2006a].

A Quick Comparison

1. Training/Preprocessing.
   - others — yes.
   - proposed — nearly none.
2. Geometry.



- others — **similarity** measures (e.g., maximum canonical correlation in MSM and sum of canonical correlations in DCC).
- proposed — classification is done on Grassmann manifold, hence Grassmannian **distances/metrics**.

By introducing the idea of Grassmannian, we are able to use many existing tools such as the Grassmannian metrics and Karcher mean to study the geometry of the data sets.

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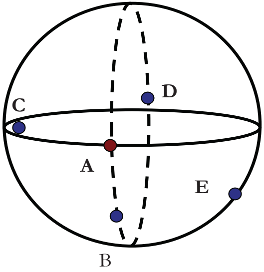

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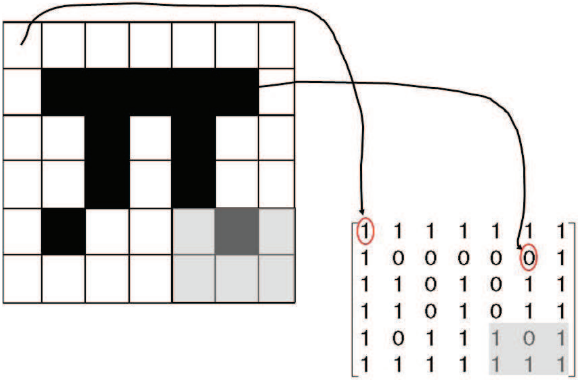
Mathematical Setup

An r-by-c gray scale digital image corresponds to an r-by-c matrix, X, where each entry enumerates one of the 256 possible gray levels of the corresponding pixel.



Mathematical Setup

Realize the data matrix, X, by its columns and concatenate columns into a single column vector, **x**.

$$
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 1 & 0 & 1 & 0 & 1 & 1 \\
1 & 1 & 0 & 1 & 0 & 1 & 1 \\
1 & 0 & 1 & 1 & 1 & 0 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1
\end{bmatrix}
\longrightarrow
\begin{bmatrix}
1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ \hline
1 \\ 0 \\ 1 \\ 1 \\ 0 \\ 1 \\ \hline
\vdots \\ \hline
1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1
\end{bmatrix}
$$

Mathematical Setup

That is,

$$X = \begin{bmatrix} \mathbf{x}_1 & | & \mathbf{x}_2 & | & \cdots & | & \mathbf{x}_c \end{bmatrix} \in \mathbb{R}^{r \times c} \quad \longrightarrow \quad \mathbf{x} = \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \vdots \\ \mathbf{x}_c \end{bmatrix} \in \mathbb{R}^{rc \times 1}$$

Thus, an image J whose matrix representation, X, can be realized as a column vector of length equaling J's resolutions.

$$\boxed{\text{Image} \rightarrow \text{Matrix} \rightarrow \text{Vector}}$$

Mathematical Setup

- Now, for a subject i, we collect k distinct images, which corresponds to k column vectors, $\mathbf{x}_j^{(i)}$ for $j = 1, 2, \ldots, k$.
- Store them into a single data matrix $X^{(i)}$ so that

$$X^{(i)} = \left[\mathbf{x}_1^{(i)} \;\middle|\; \mathbf{x}_2^{(i)} \;\middle|\; \cdots \;\middle|\; \mathbf{x}_k^{(i)} \right].$$

  Note that $\operatorname{rank}(X^{(i)}) = k$ with each $x_j^{(i)} \in \mathbb{R}^n$ being an image of resolution n.
- Associate an orthonormal basis matrix to the column space of $X^{(i)}$ (obtained via, e.g., QR or SVD), $\mathcal{R}(X^{(i)})$. Then $\mathcal{R}(X^{(i)})$ is a k-dimensional vector subspace of $\mathbb{R}^n$.

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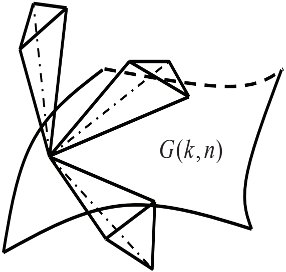
Grassmann Framework

These k-dimensional linear subspaces of $\mathbb{R}^n$ are all elements of a parameter space called the **Grassmannian (Grassmann manifold)**, $G(k,n)$, where n is the ambient resolution dimension.



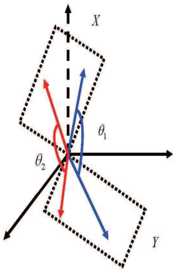
Definition

The *Grassmannian G(k,n)* or the *Grassmann manifold* is the set of k-dimensional subspaces in an n-dimensional vector space K^n for some field K, i.e.,

$$G(k,n) = \{W \subset K^n \mid \dim(W) = k\}.$$

Principal Angles [Björck & Golub, 1973]

It turns out that any attempt to construct an unitarily invariant metric on $G(k,n)$ yields something that can be expressed in terms of the **principal angles** [Stewart & Sun, 1990].



Definition

(**Principal Angles**) If X and Y are two subspaces of $\mathbb{R}^m$, then the principal angles $\theta_k \in \left[0, \frac{\pi}{2}\right]$, $1 \leq k \leq q$ between X and Y are defined recursively by

$$\cos(\theta_k) = \max_{u \in X} \max_{v \in Y} u^T v = u_k^T v_k$$

s.t. $\|u\| = \|v\| = 1$, $u^T u_i = 0$, $v^T v_i = 0$ for $i = 1, 2, \ldots, k-1$ and $q = \min\{\dim(X), \dim(Y)\} \geq 1$.

SVD-based Algorithm for Principal Angles

[Knyazev et al., 2002] For $A \in \mathbb{R}^{n\times p}$ and $B \in \mathbb{R}^{n\times q}$.

1. Find orthonormal bases Q_a and Q_b for A and B such that
$$Q_a^T Q_a = Q_b^T Q_b = I \quad \text{and} \quad \mathcal{R}(Q_a) = \mathcal{R}(A), \mathcal{R}(Q_b) = \mathcal{R}(B).$$
2. Compute SVD for cosine: $Q_a^T Q_b = Y \Sigma Z^T$, $\Sigma = \mathrm{diag}(\sigma_1, \ldots, \sigma_q)$.
3. Compute matrix
$$B = \begin{cases} Q_b - Q_a(Q_a^T Q_b) & \text{if } \mathrm{rank}(Q_a) \geq \mathrm{rank}(Q_b); \\ Q_a - Q_b(Q_b^T Q_a) & \text{otherwise.} \end{cases}$$
4. Compute SVD for sine: $[Y, \mathrm{diag}(\mu_1, \ldots, \mu_q), Z] = svd(B)$.
5. Compute the principal angles, for $k = 1, \ldots, q$:
$$\theta_k = \begin{cases} \arccos(\sigma_k) & \text{if } \sigma_k^2 < \frac{1}{2}; \\ \arcsin(\mu_k) & \text{if } \mu_k^2 \leq \frac{1}{2}. \end{cases}$$

Grassmannian Distances [Edelman et al., 1999]

Metric	**Mathematical Expression**
Fubini-Study	$d_{FS}(\mathcal{X}, \mathcal{Y}) = \cos^{-1}\left(\prod_{i=1}^{k} \cos\theta_i\right)$
Geodesic (Arc Length)	$d_g(\mathcal{X}, \mathcal{Y}) = \|\theta\|_2$
Chordal (Projection F-norm)	$d_c(\mathcal{X}, \mathcal{Y}) = \|\sin\theta\|_2$
Projection 2-norm	$d_{p2}(\mathcal{X}, \mathcal{Y}) = \|\sin\theta\|_\infty$
Chordal 2-norm	$d_{c2}(\mathcal{X}, \mathcal{Y}) = \left\|2\sin\frac{1}{2}\theta\right\|_F$
Chordal F-norm	$d_{cF}(\mathcal{X}, \mathcal{Y}) = \left\|2\sin\frac{1}{2}\theta\right\|_2$

Various Realizations of the Grassmannian

1. First, as a quotient (homogeneous space) of the orthogonal group,

$$G(k,n) = O(n)/O(k) \times O(n-k). \tag{1}$$

2. Next, as a submanifold of projective space,

$$G(k,n) \subset \mathbb{P}(\Lambda^q \mathbb{R}^n) = \mathbb{P}^{\binom{n}{k}-1}(\mathbb{R}) \tag{2}$$

via the Plücker embedding.

3. Finally, as a submanifold of Euclidean space,

$$G(k,n) \subset \mathbb{R}^{(n^2+n-2)/2} \tag{3}$$

via a projection embedding described in [Conway et al., 1996].

The Corresponding Grassmannian Distances

1. The standard invariant Riemannian metric on orthogonal matrices $O(n)$ descends via (1) to a Riemannian metric on the homogeneous space $G(k, n)$. We call the resulting geodesic distance function on the Grassmannian the *arc length* or *geodesic* distance and denote it d_g.
2. If one prefers the realization (2), then the Grassmannian inherits a Riemannian metric from the *Fubini-Study* metric on projective space (see, e.g., [Griffiths & Harris, 1978]).
3. One can restrict the usual Euclidean distance function on $\mathbb{R}^{(n^2+n-2)/2}$ to the Grassmannian via (3) to obtain the *projection F* or *chordal* distance d_c.

Grassmannian Semi-Distances

- Often time, the data set is compact and fixed.
- First few principal angles contain discriminatory information and are less sensitive to noise.
- Thus, it is natural to consider the nested subspaces. Define the ℓ-truncated principal angle vector $\theta^\ell := (\theta_1, \theta_2, \ldots, \theta_\ell)$. Then we have example ℓ-truncated Grassmannian semi-distances:

$$d_g^\ell := \|\theta^\ell\|_2, \qquad d_{FS}^\ell := \cos^{-1} \prod_{i=1}^{\ell} \cos\theta_i,$$

$$d_c^\ell := \|\sin\theta^\ell\|_2 \qquad d_{cF}^\ell := \|2\sin\tfrac{1}{2}\theta^\ell\|_2.$$

Separation Gap & Grassmann Separable

Given a set of image sets $\mathcal{P} = \{X_1, X_2, \ldots, X_m\}$, where $X_i \in \mathbb{R}^{n \times k_i}$ and each X_i belongs to one of the subject class C_j.

- Let **cardinality** of a set of images be the number of distinct images used.
- The distances between different realizations of subspaces for the same class are called **match distances** while for different classes they are called **non-match distances**.
- $W_i = \{j \mid X_j \in C_i\}$, the within-class set of subject i, and $B_i = \{j \mid X_j \notin C_i\}$, the between-class set of subject i.

Separation Gap & Grassmann Separable

- Let M **be the maximum of the match distances**

$$M = \max_{1\le i\le m} \max_{j\in W_i} d(X_i, X_j)$$

  and m **be the minimum of the non-match distances**

$$m = \min_{1\le i\le m} \min_{k\in B_i} d(X_i, X_k),$$

  then define the **separation gap** to be $g_s = m - M$.

- Then we say the set $\mathcal{P}$ is **Grassmann separable** if the separation gap is positive. i.e.,

$$\boxed{g_s > 0 \Leftrightarrow \mathcal{P} \text{ is Grassmann separable}}$$

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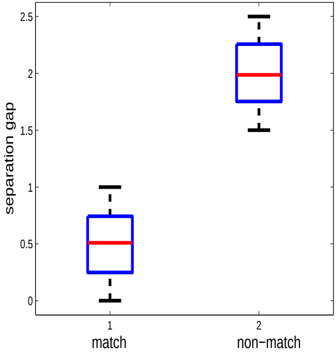
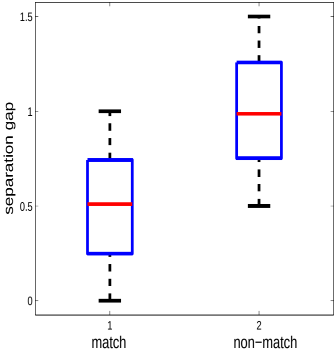
A Graphical Example



Grassmann separable



Non-Grassmann separable

Measure of Classification Rates

- **False accept rate (FAR)** is the ratio of the number of false acceptances divided by the number of identification attempts.
- **False reject rate (FRR)** is the ratio of the number of false rejections divided by the number of identification attempts.
- Given match and non-match distances for a set of classes, the **false accept rate (FAR) at a zero false reject rate (FRR)** (defined, e.g., in [Mansfield & Wayman, 2002]) is **the ratio of the number of non-match distances that are smaller than the maximum of the match distances divided by the number of non-match distances.**

$$\boxed{\text{zero percent FAR at a zero FRR} \Longleftrightarrow g_s > 0}$$

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Empirical fact

Images of a single person seen under variations of illumination appear to be more difficult to recognize than images of different people [Zhao et al., 2003].

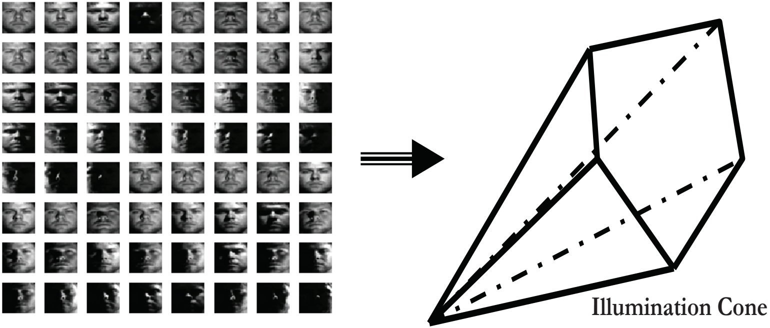
Subject 1

Subject 2

Can you tell who this is?

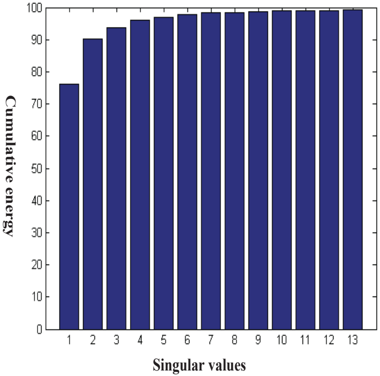
Geometric facts - 1

The set of m-pixel monochrome images of an object seen under general lighting conditions forms a convex polyhedral cone (illumination cone) in $\mathbb{R}^m$ [Belhumeur & Kriegman, 1998].

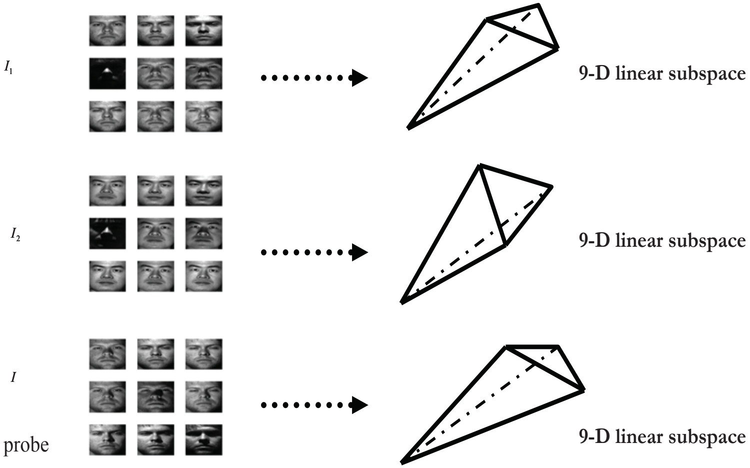


Geometric facts - 2

The illumination cone can be approximated by a 9-dimensional linear subspace [Basri & Jacobs, 2003], i.e., the illumination cone is low-dimensional and linear.

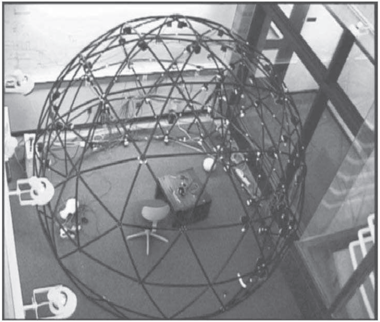


Grassmann Set-up

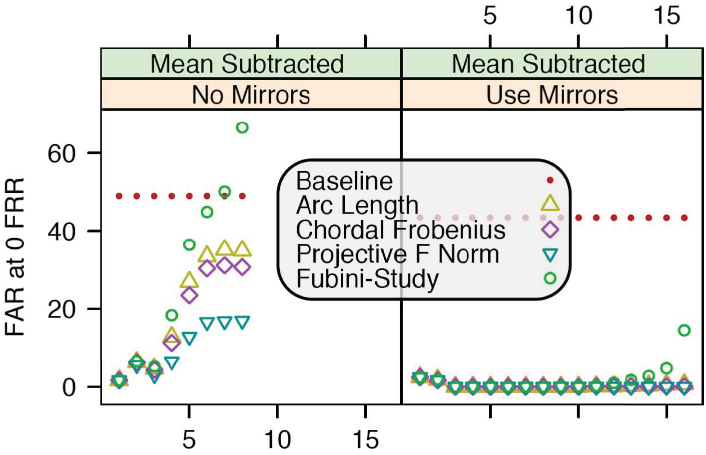


Yale Face Database B (YDB)

10 subjects, 64 illumination conditions, 9 poses



Classification Result [Chang et al., 2006a]

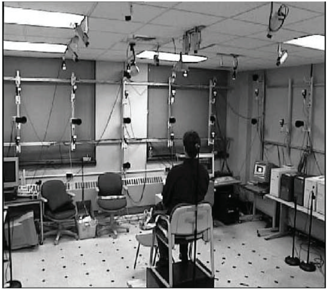


CMU-PIE

We fix the frontal pose, neutral expression and select the "illum" and "lights" subsets of CMU-PIE (68 subjects, 13 poses, 43 lightings, 4 expressions) [Sim et al., 2003] for experiments.

(a) lights: 21 illumination conditions with background lights **on**.

(b) illum: 21 illumination conditions with background lights **off**.



(a) "lights" subset

(b) "illum" subset

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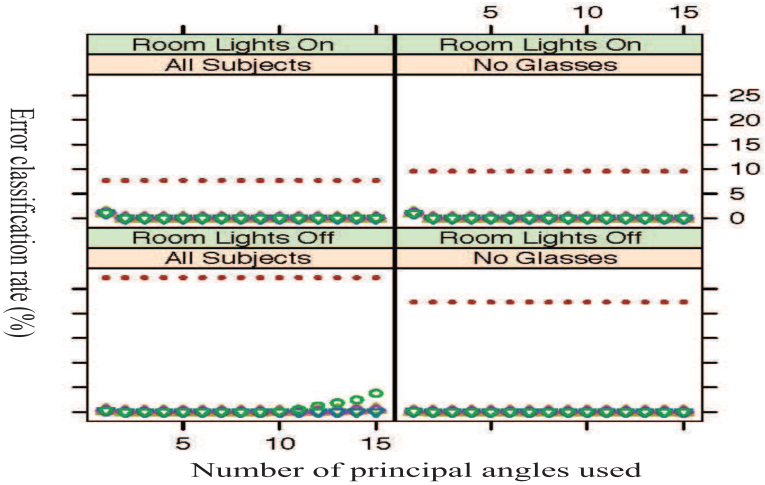
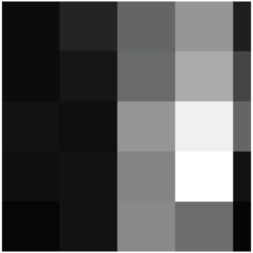


Patch Collapsing [Chang et al., 2007b]

If the data set is Grassmann separable using subject illumination subspaces of this kind of image [Chang et al., 2006a]:

The data set is still Grassmann separable using subject illumination subspaces of this kind of image [Chang et al., 2007b]:



Patch Projection [Chang et al., 2007c]

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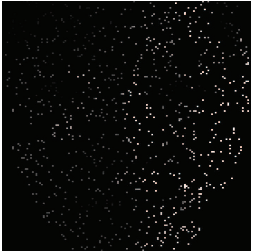

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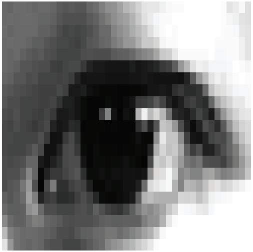

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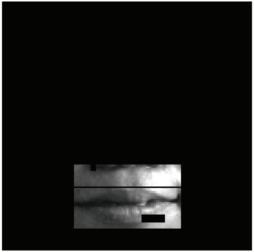
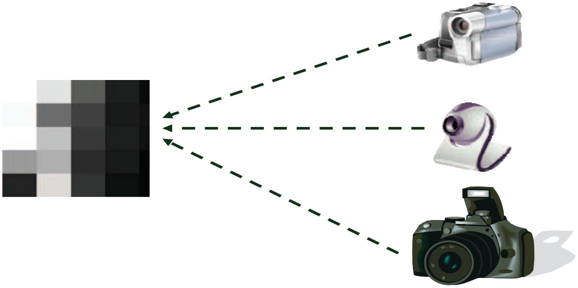


Potential Use: Low-Res. Illumination Camera



Large private databases of facial imagery can be stored at a resolution that is sufficiently low to prevent recognition by a human operator yet sufficiently high to enable machine recognition.

Karcher Mean

- How should we choose subject subspace representations given a set of images?
- Patch collapsing (e.g., low res. images) and projections (e.g., lip and nose feature patches) provide one way of compression. In particular, compression in n for points in $G(k,n)$.
- What about compression in the other parameter, k?
- To this end, we will use another geometric concept, Karcher mean, on the Grassmann manifold to accomplish this.

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Notions of Mean

- For a set of points $\{x^{(1)}, x^{(2)}, \ldots, x^{(P)}\} \in \mathbb{R}^n$, its Euclidean mean is the x that minimizes the sum squared distance

$$\sum_{i=1}^{P} d^2\left(x - x^{(i)}\right),$$

where d is the straight-line distance defined by the vector 2-norm.

- Given the points $p_1, \ldots, p_m \in G(k,n)$, the Karcher mean is the point q^* that minimizes the sum of the squares of the geodesic distance between q^* and p_i's, i.e.,

$$q^* = \arg\min_{q \in G(k,n)} \frac{1}{2m} \sum_{j=1}^{m} d^2(q, p_j),$$

where $d(q,p)$ is the geodesic distance between p and q on $G(k,n)$.

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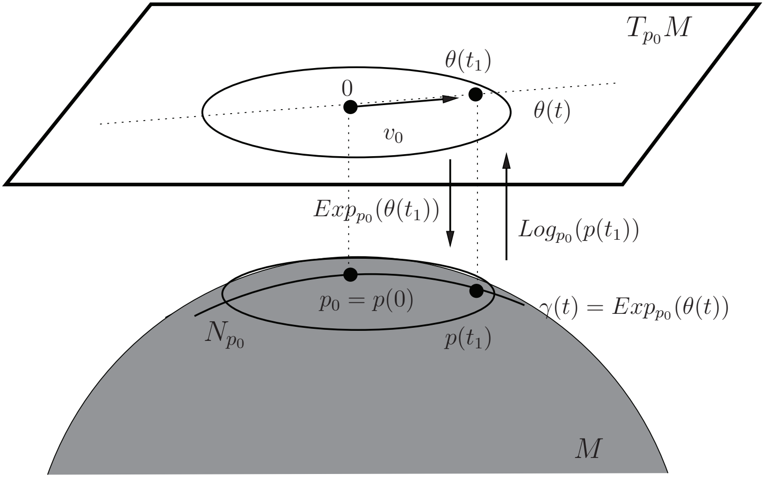
Descent Algorithm [Rahman et al., 2005]



An SVD-based Algorithm [Begelfor & Werman, 2003]

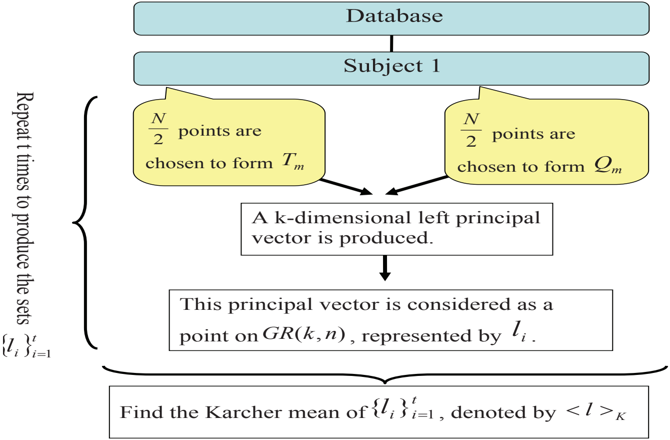
For points $p_1, p_2, \ldots, p_m \in G(k,n)$ and ϵ (machine zero), find the Karcher mean, q.

1. Set $q = p_1$.
2. Find
$$A = \frac{1}{m}\sum_{i=1}^{m} \mathrm{Log}_q(p_i).$$
3. If $||A|| < \epsilon$, return q, else, go to step 4.
4. Find the SVD $U\Sigma V^T = A$ and update
$$q \to qV\cos(\Sigma) + U\sin(\Sigma).$$
Go to step 2.

Note: the map in step 4 is the *Exponential map* that takes points from the tangent space back to the manifold.

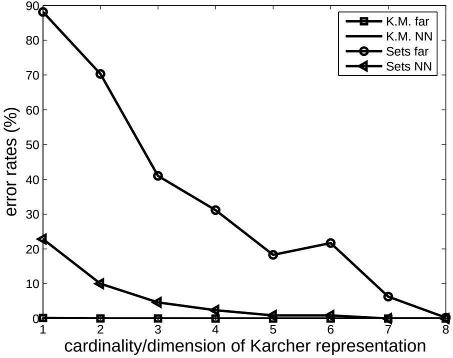
An Example Result

Compress data with k-d Karcher mean and compare the recognition result to results obtained using k raw images.



An Example Result

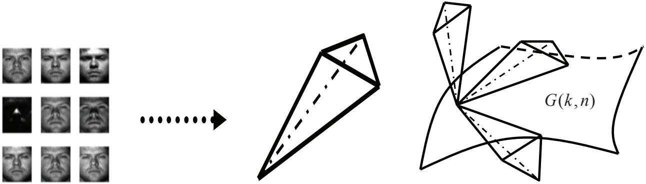
- 16 images used for gallery pts; 3 images used for probes.
- Lip patch on the CMU-PIE “lights” data set.



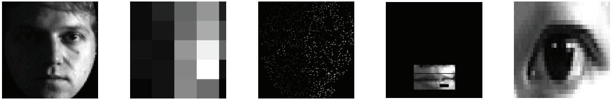
The fact that using a 1-d Karcher representation achieves a perfect recognition result while using 1 raw image in the gallery does not indicates that Karcher representations are able to pack useful information more efficiently.

Conclusions

- A novel geometric framework for a many-to-many architecture — Grassmann framework.

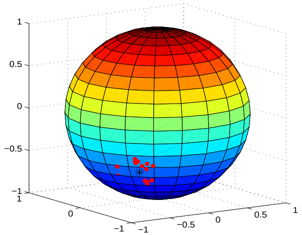


- Empirical results and new insights.



Conclusions

- A novel algorithm for Karcher compression.



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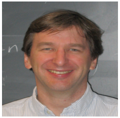

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