Classification on the Grassmannians: Theory and Applications

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Outline

1. Geometric Framework
   - Evolution of Classification Paradigms
   - Grassmann Framework
   - Grassmann Separability

2. Some Empirical Results
   - Illumination
   - Illumination + Low Resolutions

3. Compression on $G(k, n)$
   - Motivations, Definitions, and Algorithms
   - Karcher Compression for Face Recognition
Architectures

Historically

- single-to-single

\[ P = \begin{bmatrix} x^{(1)} \\ \vdots \\ x^{(N)} \end{bmatrix}, \quad G = \begin{bmatrix} d(P, x^{(1)}) \\ \vdots \\ d(P, x^{(N)}) \end{bmatrix} \]

- single-to-many

\[ P = \begin{bmatrix} x^{(1)} \\ \vdots \\ x^{(N)} \end{bmatrix}, \quad G = \begin{bmatrix} \text{Eigenfaces} \end{bmatrix} \]

Currently

- subspace-to-subspace

\[ P = \begin{bmatrix} x^{(1)} \\ \vdots \\ x^{(N)} \end{bmatrix}, \quad G = \begin{bmatrix} \text{Grassmann Manifold} \end{bmatrix} \]

- many-to-many
Some Approaches

- **Single-to-Single**
  1. Euclidean distance of feature points.
  2. Correlation.

- **Single-to-Many**
  3. Linear/Fisher Discriminate Analysis, Fisherfaces [Belhumeur et al., 1997].
Some Approaches

- Many-to-Many
  - Tangent Space and Tangent Distance - Tangent Distance [Simard et al., 2001], Joint Manifold Distance [Fitzgibbon & Zisserman, 2003], Subspace Distance [Chang, 2004].
  - Manifold Density Divergence [Fisher et al., 2005].
  - Canonical Correlation Analysis (CCA):
    - Mutual Subspace Method (MSM) [Yamaguchi et al., 1998],
    - Constrained Mutual Subspace Method (CMSM) [Fukui & Yamaguchi, 2003],
    - Multiple Constrained Mutual Subspace Method (MCMSM) [Nishiyama et al., 2005],
    - Kernel CCA [Wolf & Shashua, 2003],
    - Discriminant Canonical Correlation (DCC) [Kim et al., 2006],
    - Grassmann method [Chang et al., 2006a].
A Quick Comparison

1. Training/Preprocessing.
   - others — yes.
   - proposed — nearly none.

2. Geometry.
   - others — similarity measures (e.g., maximum canonical correlation in MSM and sum of canonical correlations in DCC).
   - proposed — classification is done on Grassmann manifold, hence Grassmannian distances/metrics.

By introducing the idea of Grassmannian, we are able to use many existing tools such as the Grassmannian metrics and Karcher mean to study the geometry of the data sets.
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Mathematical Setup

An \( r \)-by-\( c \) gray scale digital image corresponds to an \( r \)-by-\( c \) matrix, \( X \), where each entry enumerates one of the 256 possible gray levels of the corresponding pixel.
Mathematical Setup

Realize the data matrix, $X$, by its columns and concatenate columns into a single column vector, $\mathbf{x}$.
Mathematical Setup

That is,

\[ X = \begin{bmatrix} x_1 & x_2 & \cdots & x_c \end{bmatrix} \in \mathbb{R}^{r \times c} \rightarrow x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_c \end{bmatrix} \in \mathbb{R}^{rc \times 1} \]

Thus, an image \( J \) whose matrix representation, \( X \), can be realized as a column vector of length equaling \( J \)'s resolutions.

\[ \text{IMAGE} \rightarrow \text{MATRIX} \rightarrow \text{VECTOR} \]
Mathematical Setup

Now, for a subject $i$, we collect $k$ distinct images, which corresponds to $k$ column vectors, $x_j^{(i)}$ for $j = 1, 2, \ldots, k$.

Store them into a single data matrix $X^{(i)}$ so that

$$X^{(i)} = \begin{bmatrix} x_1^{(i)} \mid x_2^{(i)} \mid \cdots \mid x_k^{(i)} \end{bmatrix}.$$

Note that $\text{rank}(X^{(i)}) = k$ with each $x_j^{(i)} \in \mathbb{R}^n$ being an image of resolution $n$.

Associate an orthonormal basis matrix to the column space of $X^{(i)}$ (obtained via, e.g., QR or SVD), $\mathcal{R}(X^{(i)})$. Then $\mathcal{R}(X^{(i)})$ is a $k$-dimensional vector subspace of $\mathbb{R}^n$. 
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Grassmann Framework

These $k$-dimensional linear subspaces of $\mathbb{R}^n$ are all elements of a parameter space called the **Grassmannian (Grassmann manifold)**, $G(k, n)$, where $n$ is the ambient resolution dimension.

**Definition**

The **Grassmannian** $G(k, n)$ or the **Grassmann manifold** is the set of $k$-dimensional subspaces in an $n$-dimensional vector space $K^n$ for some field $K$, i.e.,

$$G(k, n) = \{ W \subset K^n \mid \dim(W) = k \}.$$
Principal Angles [Björck & Golub, 1973]

It turns out that any attempt to construct an unitarily invariant metric on $G(k, n)$ yields something that can be expressed in terms of the principal angles [Stewart & Sun, 1990].

**Definition**

*(Principal Angles)* If $X$ and $Y$ are two subspaces of $\mathbb{R}^m$, then the principal angles $\theta_k \in \left[0, \frac{\pi}{2}\right]$, $1 \leq k \leq q$ between $X$ and $Y$ are defined recursively by

$$
\cos(\theta_k) = \max_{u \in X} \max_{v \in Y} u^T v = u_k^T v_k
$$

s.t. $\|u\| = \|v\| = 1$, $u^T u_i = 0$, $v^T v_i = 0$ for $i = 1, 2, \ldots, k - 1$ and $q = \min \{\dim(X), \dim(Y)\} \geq 1.$
SVD-based Algorithm for Principal Angles

[Knyazev et al., 2002] For $A \in \mathbb{R}^{n \times p}$ and $B \in \mathbb{R}^{n \times q}$.

1. Find orthonormal bases $Q_a$ and $Q_b$ for $A$ and $B$ such that

$$Q_a^T Q_a = Q_b^T Q_b = I \quad \text{and} \quad \mathcal{R}(Q_a) = \mathcal{R}(A), \mathcal{R}(Q_b) = \mathcal{R}(B).$$

2. Compute SVD for cosine: $Q_a^T Q_b = Y \Sigma Z^T$, $\Sigma = \text{diag}(\sigma_1, \ldots, \sigma_q)$.

3. Compute matrix

$$B = \begin{cases} Q_b - Q_a (Q_a^T Q_b) & \text{if rank}(Q_a) \geq \text{rank}(Q_b); \\ Q_a - Q_b (Q_b^T Q_a) & \text{otherwise}. \end{cases}$$

4. Compute SVD for sine: $[Y, \text{diag}(\mu_1, \ldots, \mu_q), Z] = \text{svd}(B)$.

5. Compute the principal angles, for $k = 1, \ldots, q$:

$$\theta_k = \begin{cases} \arccos(\sigma_k) & \text{if } \sigma_k^2 < \frac{1}{2}; \\ \arcsin(\mu_k) & \text{if } \mu_k^2 \leq \frac{1}{2}. \end{cases}$$
Grassmannian Distances [Edelman et al., 1999]

<table>
<thead>
<tr>
<th>Metric</th>
<th>Mathematical Expression</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fubini-Study</td>
<td>$d_{FS} (\mathcal{X}, \mathcal{Y}) = \cos^{-1} \left( \prod_{i=1}^{k} \cos \theta_i \right)$</td>
</tr>
<tr>
<td>Geodesic (Arc Length)</td>
<td>$d_g (\mathcal{X}, \mathcal{Y}) = | \theta |_2$</td>
</tr>
<tr>
<td>Chordal (Projection F-norm)</td>
<td>$d_c (\mathcal{X}, \mathcal{Y}) = | \sin \theta |_2$</td>
</tr>
<tr>
<td>Projection 2-norm</td>
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</table>
Various Realizations of the Grassmannian

1. First, as a quotient (homogeneous space) of the orthogonal group,

\[ G(k, n) = O(n) / O(k) \times O(n-k). \]  \hspace{1cm} (1)

2. Next, as a submanifold of projective space,

\[ G(k, n) \subset \mathbb{P}(\Lambda^q \mathbb{R}^n) = \mathbb{P}^n_x^{-1}(\mathbb{R}) \] \hspace{1cm} (2)

via the Plücker embedding.

3. Finally, as a submanifold of Euclidean space,

\[ G(k, n) \subset \mathbb{R}^{(n^2+n-2)/2} \] \hspace{1cm} (3)

via a projection embedding described in [Conway et al., 1996].
The Corresponding Grassmannian Distances

1. The standard invariant Riemannian metric on orthogonal matrices $O(n)$ descends via (1) to a Riemannian metric on the homogeneous space $G(k, n)$. We call the resulting geodesic distance function on the Grassmannian the *arc length* or *geodesic* distance and denote it $d_g$.

2. If one prefers the realization (2), then the Grassmannian inherits a Riemannian metric from the *Fubini-Study* metric on projective space (see, e.g., [Griffiths & Harris, 1978]).

3. One can restrict the usual Euclidean distance function on $\mathbb{R}^{(n^2+n-2)/2}$ to the Grassmannian via (3) to obtain the *projection* $F$ or *chordal* distance $d_c$. 
Grassmannian Semi-Distances

- Often time, the data set is compact and fixed.
- First few principal angles contain discriminatory information and are less sensitive to noise.
- Thus, it is natural to consider the nested subspaces. Define the $\ell$-truncated principal angle vector $\theta^\ell := (\theta_1, \theta_2, \ldots, \theta_\ell)$. Then we have example $\ell$-truncated Grassmannian semi-distances:

\[
\begin{align*}
    d^\ell_g &:= \|\theta^\ell\|_2, \\
    d^\ell_{FS} &:= \cos^{-1} \prod_{i=1}^{\ell} \cos \theta_i, \\
    d^\ell_c &:= \|\sin \theta^\ell\|_2, \\
    d^\ell_{CF} &:= \|2 \sin \frac{1}{2} \theta^\ell\|_2.
\end{align*}
\]
Separation Gap & Grassmann Separable

Given a set of image sets \( \mathcal{P} = \{X_1, X_2, \ldots, X_m\} \), where \( X_i \in \mathbb{R}^{n \times k_i} \) and each \( X_i \) belongs to one of the subject class \( C_j \).

- Let **cardinality** of a set of images be the number of distinct images used.

- The distances between different realizations of subspaces for the same class are called **match distances** while for different classes they are called **non-match distances**.

- \( W_i = \{j \mid X_j \in C_i\} \), the within-class set of subject \( i \), and \( B_i = \{j \mid X_j \notin C_i\} \), the between-class set of subject \( i \).
Separation Gap & Grassmann Separable

- Let $M$ be the maximum of the match distances

$$M = \max_{1 \leq i \leq m} \max_{j \in W_i} d(X_i, X_j)$$

and $m$ be the minimum of the non-match distances

$$m = \min_{1 \leq i \leq m} \min_{k \in B_i} d(X_i, X_k),$$

then define the separation gap to be $g_s = m - M$.

- Then we say the set $\mathcal{P}$ is Grassmann separable if the separation gap is positive. i.e.,

$$g_s > 0 \iff \mathcal{P} \text{ is Grassmann separable}$$
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A Graphical Example

Grassmann separable

Non-Grassmann separable
Measure of Classification Rates

- **False accept rate (FAR)** is the ratio of the number of false acceptances divided by the number of identification attempts.

- **False reject rate (FRR)** is the ratio of the number of false rejections divided by the number of identification attempts.

- Given match and non-match distances for a set of classes, the **false accept rate (FAR) at a zero false reject rate (FRR)** (defined, e.g., in [Mansfield & Wayman, 2002]) is the ratio of the number of non-match distances that are smaller than the maximum of the match distances divided by the number of non-match distances.

  \[ \text{zero percent FAR at a zero FRR} \iff g_s > 0 \]
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Empirical fact

Images of a single person seen under variations of illumination appear to be more difficult to recognize than images of different people [Zhao et al., 2003].

Can you tell who this is?

Subject 1  Subject 2
The set of $m$-pixel monochrome images of an object seen under general lighting conditions forms a convex polyhedral cone (illumination cone) in $\mathbb{R}^m$ [Belhumeur & Kriegman, 1998].
The illumination cone can be approximated by a 9-dimensional linear subspace [Basri & Jacobs, 2003], i.e., the illumination cone is low-dimensional and linear.
Grassmann Set-up

\( I_1 \)

\( I_2 \)

\( I \)

probes

\[ \rightarrow \]

9-D linear subspace

\[ \rightarrow \]

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Yale Face Database B (YDB)

10 subjects, 64 illumination conditions, 9 poses
Classification Result [Chang et al., 2006a]
We fix the frontal pose, neutral expression and select the “illum” and “lights” subsets of CMU-PIE (68 subjects, 13 poses, 43 lightings, 4 expressions) [Sim et al., 2003] for experiments.

(a) lights: 21 illumination conditions with background lights on.
(b) illum: 21 illumination conditions with background lights off.
Classification Result [Chang et al., 2006a]
Patch Collapsing [Chang et al., 2007b]

If the data set is Grassmann separable using subject illumination subspaces of this kind of image [Chang et al., 2006a]:

The data set is still Grassmann separable using subject illumination subspaces of this kind of image [Chang et al., 2007b]:

![Image](image1.png)

![Image](image2.png)
Patch Projection [Chang et al., 2007c]

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Large private databases of facial imagery can be stored at a resolution that is sufficiently low to prevent recognition by a human operator yet sufficiently high to enable machine recognition.
How should we choose subject subspace representations given a set of images?

Patch collapsing (e.g., low res. images) and projections (e.g., lip and nose feature patches) provide one way of compression. In particular, compression in $n$ for points in $G(k, n)$.

What about compression in the other parameter, $k$?

To this end, we will use another geometric concept, Karcher mean, on the Grassmann manifold to accomplish this.
Karcher Mean

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Notions of Mean

For a set of points $\{x^{(1)}, x^{(2)}, \ldots, x^{(P)}\} \in \mathbb{R}^n$, its Euclidean mean is the $x$ that minimizes the sum squared distance

$$
\sum_{i=1}^{P} d^2 \left( x - x^{(i)} \right),
$$

where $d$ is the straight-line distance defined by the vector 2-norm.

Given the points $p_1, \ldots, p_m \in G(k, n)$, the Karcher mean is the point $q^*$ that minimizes the sum of the squares of the geodesic distance between $q^*$ and $p_i$'s, i.e.,

$$
q^* = \arg \min_{q \in G(k, n)} \frac{1}{2m} \sum_{j=1}^{m} d^2(q, p_j),
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where $d(q, p)$ is the geodesic distance between $p$ and $q$ on $G(k, n)$. 
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Descent Algorithm [Rahman et al., 2005]
For points $p_1, p_2, \ldots, p_m \in G(k, n)$ and $\epsilon$ (machine zero), find the Karcher mean, $q$.

1. Set $q = p_1$.
2. Find
   \[
   A = \frac{1}{m} \sum_{i=1}^{m} \text{Log}_q(p_i).
   \]
3. If $\|A\| < \epsilon$, return $q$, else, go to step 4.
4. Find the SVD $U \Sigma V^T = A$ and update
   \[
   q \rightarrow q V \cos(\Sigma) + U \sin(\Sigma).
   \]
   Go to step 2.

Note: the map in step 4 is the Exponential map that takes points from the tangent space back to the manifold.
An Example Result

Compress data with $k$-d Karcher mean and compare the recognition result to results obtained using $k$ raw images.

Repeat $t$ times to produce the sets $\{l_i\}_{i=1}^t$.

For each set, choose $N/2$ points to form $T_m$ and $Q_m$.

A $k$-dimensional left principal vector is produced.

This principal vector is considered as a point on $GR(k,n)$, represented by $l_i$.

Find the Karcher mean of $\{l_i\}_{i=1}^t$, denoted by $<l>_K$. 
An Example Result

- 16 images used for gallery pts; 3 images used for probes.
- Lip patch on the CMU-PIE “lights” data set.

The fact that using a 1-d Karcher representation achieves a perfect recognition result while using 1 raw image in the gallery does not indicates that Karcher representations are able to pack useful information more efficiently.
Conclusions

- A novel geometric framework for a many-to-many architecture — Grassmann framework.

- Empirical results and new insights.
Conclusions

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Selected References


