Applying Image Processing Techniques to Promote Conceptual Understanding in Linear Algebra Classes

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Outline

1. Background
2. Examples
   - Matrix multiplications vs. edge detection
   - Vector space vs. face images under lighting
   - Orthogonal projection vs. novelty filter
3. Evaluation
4. Student Feedbacks
5. Challenges and Future Work

Institution Background

- CSULB is one of the 23 campuses comprising the California State University system.
- CSULB is second only to UCLA as the largest university in California ($\approx$ 38,000 student population). In the nation, only UCLA and UC Berkeley received more applications ($\approx$ 74,000 applications received in 2009 AY).
- The mission of CSULB is to provide affordable high quality education and valued baccalaureate and master's degrees in a broad range of basic, applied, and professional fields.

Course Background

M247: Introduction to Linear Algebra

Spring 2009		Spring 2010	
Major	*#/41*	*Major*	*#/33*
Aero. Eng.	1	Chemistry	1
Civil Eng.	1	Chem. Eng.	1
Mech. Eng.	1	Civil Eng.	1
Chemistry	2	Mech. Eng.	1
Elec. Eng.	2	Physics	1
Physics	3	Economics	2
CS	7	Math	2
Math Ed	7	Math Ed	8
Math	10	CS	16
Others[1]	7		

[1] Post-Bac, art, single subject credential, accountancy, undeclared, liberal studies

Motivations - A Practical Need

- The impact of CSULB on the regional economy is estimated to be of the order of 1 billion annually. More than 2 billion of the earnings by alumni from CSULB are attributable to their CSU degree.
- Long Beach derives its economic security from the aerospace, harbor, oil, and tourism industries, all of which requires mathematics in a practical setting.

Motivations - A Request From Interdisciplinary Partners

When a student is required to take a course outside of the discipline for his or her major, some effort should be made to help the student understand why that topic is important for the major. This can make the difference between a student valuing the course as something they want to remember and learn more of and one who cannot wait to sell the textbook after the semester. This becomes even more important when the course is not a central part of the discipline. — Dr. Todd Ebert, Associate Professor of CECS at CSULB.

Goal

Unlike calculus and other topics, we do not have data about failure rates or attrition, analysis of exam questions and results, or documentation of complaints from faculty who teach course for which linear algebra is a prerequisite. — Dubinsky, 1997

- Ultimately, we are interested in how students learn and perceive concepts in linear algebra.
- To this end, we adopt a motivation-first, theory-second approach to stimulate students' intellectual needs for learning.

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Before ...

- *I thought Linear Algebra would be a VERY difficult course to take.*
- *I thought it was going to be an incredibly challenging class and I didn't think I would be able to pass on my first attempt.*
- *I was under the impression that linear algebra was extremely difficult compared to say Calculus. It didn't help that people I know who were math majors struggled with the class as well.*
- *I always thought that it's a hard class.*
- *I didn't think linear algebra could go deeper than beyond matrices.*

- *Before taking the class, I didn't know how many applications of linear algebra exist.*

Objectives: (1) to understand the physical meaning of matrix multiplication and what it does to arrays of information. (2) to see that matrix multiplication is noncommutative.

Let X be the 360×360 matrix (why?) that represents the image here.

X has integer entries ranging from 0 to 255 (why?) and is of the form

$$X = \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1,360} \\ x_{21} & x_{22} & \cdots & x_{2,360} \\ \vdots & \vdots & \ddots & \vdots \\ x_{360,1} & x_{360,2} & \cdots & x_{360,360} \end{bmatrix}.$$

Consider another 360×360 matrix

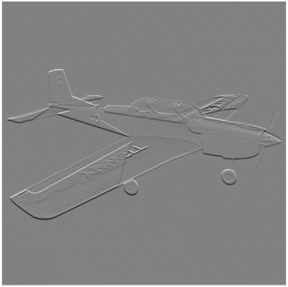
$$H = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & -1 \\ -1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 \end{bmatrix}.$$

We ask: What effect does H have on entries of X? what happens when we compute HX?

$$
\begin{aligned}
HX &= \begin{bmatrix} 1 & -1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & -1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & -1 \\ -1 & 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1,360} \\ x_{21} & x_{22} & \cdots & x_{2,360} \\ \vdots & \vdots & \ddots & \vdots \\ x_{360,1} & x_{360,2} & \cdots & x_{360,360} \end{bmatrix} \\
&= \begin{bmatrix} x_{11} - x_{21} & x_{12} - x_{22} & \cdots & x_{1,360} - x_{2,360} \\ x_{21} - x_{31} & x_{22} - x_{32} & \cdots & x_{2,360} - x_{3,360} \\ \vdots & \vdots & \ddots & \vdots \\ x_{360,1} - x_{11} & x_{360,2} - x_{12} & \cdots & x_{360,360} - x_{1,360} \end{bmatrix}.
\end{aligned}
$$

We ask: What's the result of this product?

- In areas where the pixels in consecutive rows do not change much in value, the components of HX are pretty close to zero (black).
- Whenever there is a boundary along a row, the values of HX will be large (white), and the larger they are the whiter the resulting pixel.
- So left multiplying by H can be thought of as a method for detecting boundaries along the rows in X (a.k.a. horizontal edge detector).



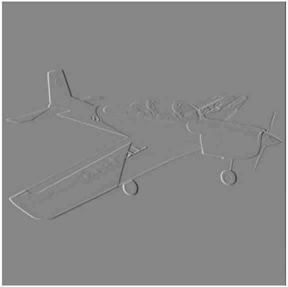
We ask: How do we compute differences along the consecutive columns of X?

Consider the product XH^T:

$$XH^T = \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1,360} \\ x_{21} & x_{22} & \cdots & x_{2,360} \\ \vdots & \vdots & \ddots & \vdots \\ x_{360,1} & x_{360,2} & \cdots & x_{360,360} \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & \cdots & 0 & -1 \\ -1 & 1 & \cdots & 0 & 0 \\ 0 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} x_{11} - x_{12} & x_{12} - x_{13} & \cdots & x_{1,360} - x_{11} \\ x_{21} - x_{22} & x_{22} - x_{23} & \cdots & x_{2,360} - x_{21} \\ \vdots & \vdots & \ddots & \vdots \\ x_{360,1} - x_{360,2} & x_{360,2} - x_{360,3} & \cdots & x_{360,360} - x_{360,1} \end{bmatrix}.$$

Similarly, right multiplying X by H^T can be thought of as a method for detecting boundaries along the columns in X (a.k.a. vertical edge detector).



We ask: (1) What does XH look like? (2) Do you think $XY = YX$ for any two compatible matrices X and Y in general? (3) How many of you would love to see a side-by-side comparison now?

Objective: to motivate the axiomatic definition of vector space through the use of face images under variation of illumination. Ideally students will see why these properties are needed.

For $u, v, w \in V$, $c, d \in \mathbb{R}$. Denote

u

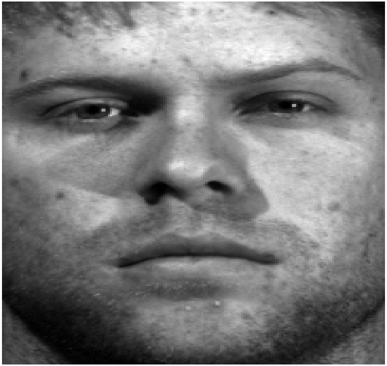
v

w

VS - $u + v \in V$

Property 1: sum of any two images of the same person under two (possibly different) lighting conditions is again a face image of the same person under some lighting condition.

u + v



VS - $u + v = v + u$

Property 2: sum of any two images of the same person under two (possibly different) lighting conditions is independent of the order of addition.

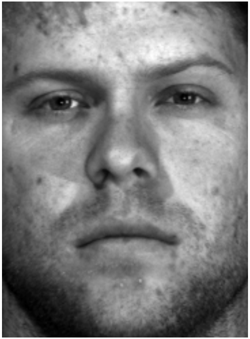
u + v

v + u

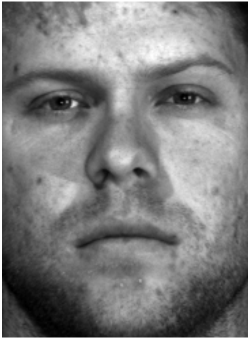
VS - $(u+v)+w = u+(v+w)$

Property 3: sum of any three images of the same person under three (possibly different) lighting conditions, taken two at a time, is independent of the order.

(u + v) + w



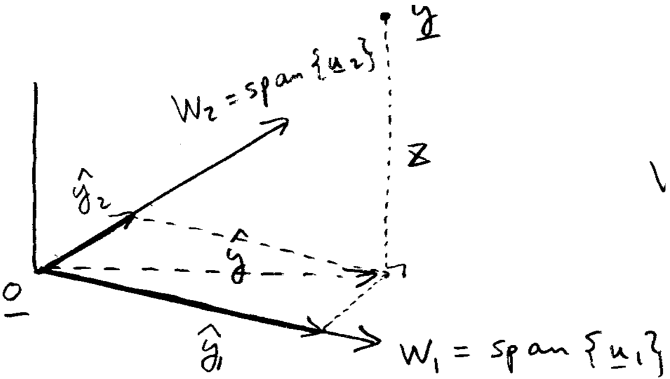
u + (v + w)



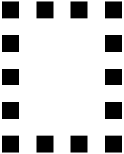
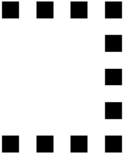
Objective: to understand the mathematical meaning and practical usage of the equation

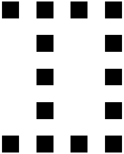
$$y = \hat{y} + z, \quad \text{where} \quad \hat{y} = \frac{y^T u_1}{u_1^T u_1} u_1 + \cdots + \frac{y^T u_p}{u_p^T u_p} u_p.$$

$\hat{y}$ is called the *orthogonal projection* of y unto W with orthogonal basis $\{u_1, \cdots, u_p\}$ and $z \in W^{\perp}$ is called the *residual* of the projection.



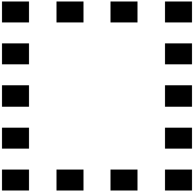
Imagine the following 5×4 arrays of black squares in $\mathbb{R}^{20}$ (why?) with

$\mathbf{v}_1 =$  $\mathbf{v}_2 =$  $\mathbf{v}_3 =$ 

$\mathbf{y} =$ 

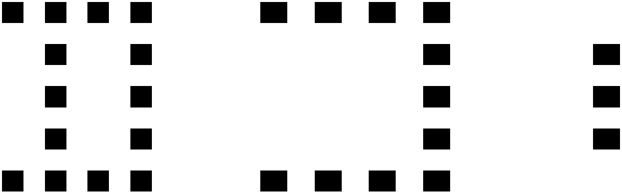
We ask: *How is* $\mathbf{y}$ *different from the space spanned by* $\mathbf{v}_1, \mathbf{v}_2$*, and* $\mathbf{v}_3$*? i.e., what is the novelty (or residue) when you orthogonally project* $\mathbf{y}$ *onto* $W = \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$*?*

Step 1: find an **orthogonal basis** for $W = \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, call it $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ (why?).
Step 2: find the orthogonal projection of $\mathbf{y}$ onto each $W_i = \text{span}\{\mathbf{u}_i\}$ for each i, denoted by $\hat{\mathbf{y}}_i$. The result of the projections is shown below.

$\hat{\mathbf{y}}_1 =$  $\hat{\mathbf{y}}_2 =$  $\hat{\mathbf{y}}_3 =$ 

where $\hat{\mathbf{y}}_i = \dfrac{\mathbf{y}^T\mathbf{u}_i}{\mathbf{u}_i^T\mathbf{u}_i}\mathbf{u}_i$ for each $i = 1, 2, 3$ (what does each $\hat{\mathbf{y}}_i$ mean?).
And $\hat{\mathbf{y}} = \hat{\mathbf{y}}_1 + \hat{\mathbf{y}}_2 + \hat{\mathbf{y}}_3$. (why?)

We can then write $\mathbf{y}$ as a sum of orthogonal vectors, one in W and one in the orthogonal complement of W, $W^{\perp}$, i.e., $\mathbf{y} = \hat{\mathbf{y}} + \mathbf{z}$, where $\hat{\mathbf{y}} \in W$ and $\mathbf{z} \in W^{\perp}$ (why and when can we do that?). See below for this graphical result.



$$\mathbf{y} \quad = \quad \hat{\mathbf{y}} \quad + \quad \mathbf{z}$$

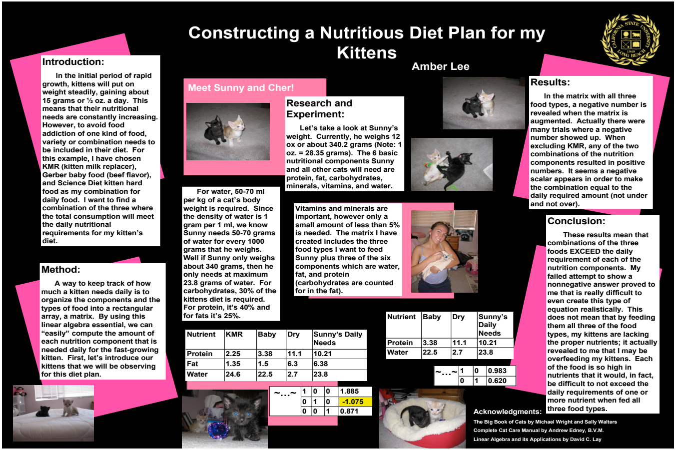
Result: the three vertical squares in the second column of $\mathbf{y}$ *appears to be different (novel) from the space* W*, hence cannot be generated from ANY linear combination of vectors in* W*.*

Poster Projects by Students

- **Matrix operations:** encoding and decoding secret messages with Hill Ciphers; applications to airline flights; relationships between individuals in a group.
- **System of equations:** constructing a nutritious diet plan for my kittens; games of strategy; constructing curves and surfaces through specified points.
- **Linear transformations:** applications in computer graphics; image compression; fractal.
- **Eigenvalue and eigenvectors:** matrix diagonalization and Fibonacci numbers; biological molecules: PCA Analysis of holo-Myoglobin; modeling physical systems: coupled differential equations.

Posters available on website:
http://www.csulb.edu/∼jchang9/m247studentPosterProjs.htm

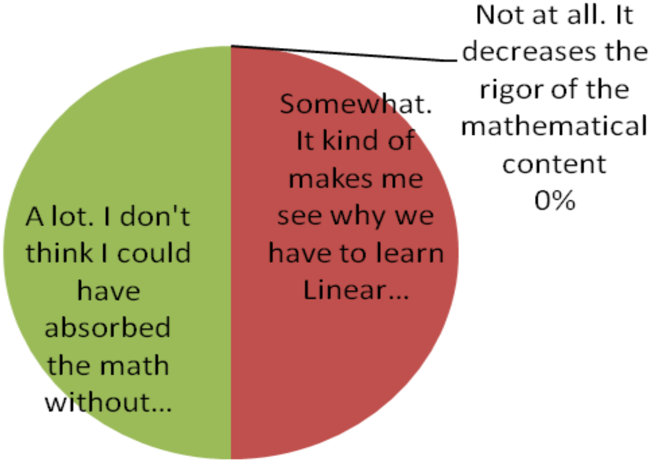
I actually found the poster project to be the most entertaining and useful assignment I had all semester. You gave us creativity and encouraged us to stretch ourselves in the mathematics field.



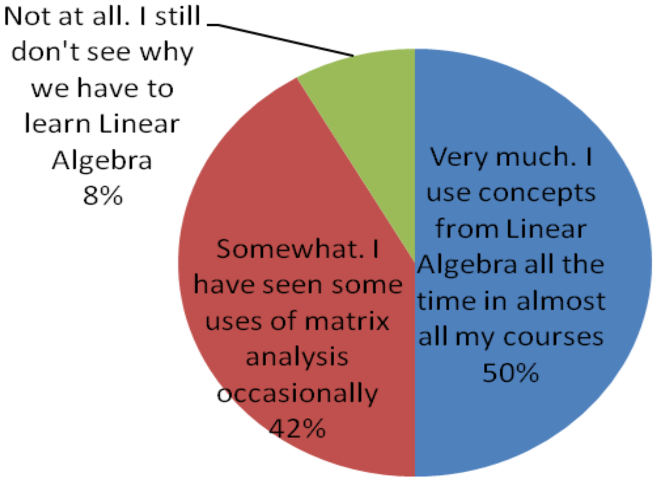
After ...

- *Over time I have started to see that it is less difficult than I originally thought.*
- *I am to be getting a good enough grade to pass the class without fear of failing.*
- *My overall perception of it has changed though ... I just felt like I was eased into it ...*
- *Now that I'm taking it for the second time I understand more about the abstract side of it.*
- *I was clearly proved wrong.*
- *Now I am even looking forward to doing the PowerPoint presentation.*

How much did you like the application aspect of the course?



How useful have you found Linear Algebra to be in your other courses?



- Building a learning community on campus. (lesson study for linear algebra)

  > *I think the image processing examples used in class are helpful, but only for certain concepts. I would personally like to see examples from other disciplines. A variety of examples prevents one from developing too narrow a context.*

- Experiment with MATLAB programming.

  > *I think it would help us learn the difficult concepts better by having a computer/numerical component because we would literally apply matrices to real-life applications. That would be beneficial as well as interesting. As long as it is done in class because if it is an added assignment outside of the class it would just be inconvenient and painful for busy college students.*

Questions?

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