An Introduction to Geometric Data Analysis and its Possible Applications

JEN-MEI CHANG

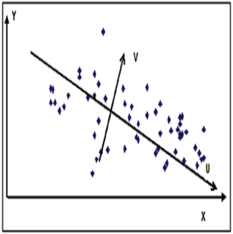
Department of Mathematics and Statistics
California State University, Long Beach
jchang9@csulb.edu

Cal State Fullerton Colloquium

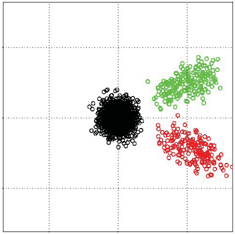
Outline

1. Introduction
   - Analysis
   - Synthesis
2. Backgrounds
   - Linear Algebra
   - Geometry
   - Image Processing
3. Applications
   - Image Compression
   - Digit/Face Recognition with Tangent Distance
   - Face Recognition on $G(k, n)$
   - Missing Data
   - Noisy Data
   - Others

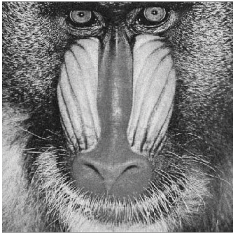
Why analysis?



Representation

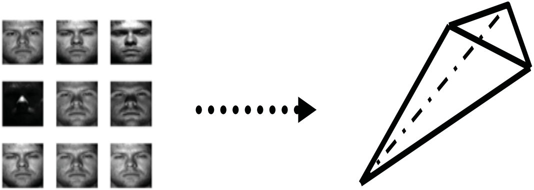


Visualization

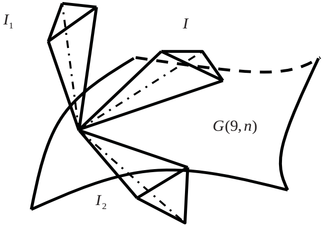


Applications

Why synthesis?



Model building



Prediction and classification

Full SVD

Definition

(**Full SVD**) Any $m \times n$ real matrix A, with $m \geq n$, can be factorized into

$$A = U \begin{pmatrix} \Sigma \\ 0 \end{pmatrix} V^T,$$

where $U \in \mathbb{R}^{m \times m}$ and $V \in \mathbb{R}^{n \times n}$ are orthogonal, and $\Sigma \in \mathbb{R}^{n \times n}$ is diagonal with

$$\Sigma = \text{diag}(\sigma_1, \sigma_2, \ldots, \sigma_n), \ \sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n \geq 0.$$

Thin SVD

Definition

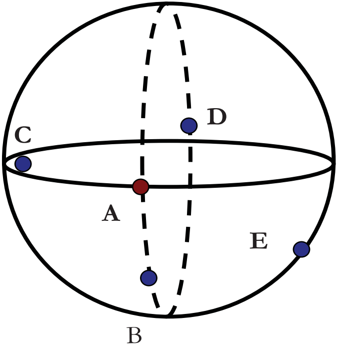
(**Thin SVD**) With the partitioning $U = (U_1, U_2)$, where $U_1 \in \mathbb{R}^{m \times n}$, we get the *thin SVD*

$$A = U_1 \Sigma V^T,$$

Structural Illustration:

$$A = U_1 \Sigma V^T = (u_1 \; u_2 \; \cdots \; u_n) \begin{pmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_n \end{pmatrix} \begin{pmatrix} v_1^T \\ v_2^T \\ \vdots \\ v_n^T \end{pmatrix} = \sum_{i=1}^{n} \sigma_i u_i v_i^T.$$

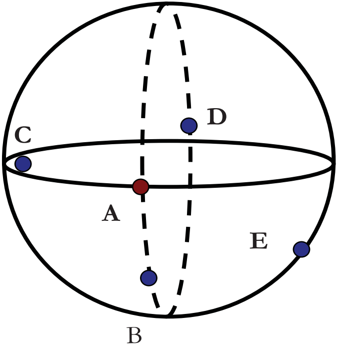
Distance



What is A closest to?

- No geometry: D
- With geometry: B

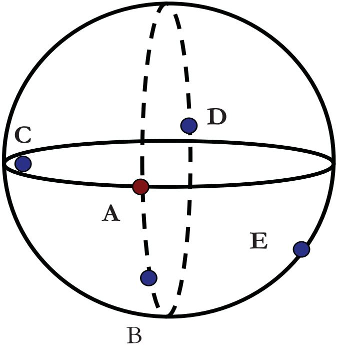
Distance



What is A closest to?

- No geometry: D
- With geometry: B

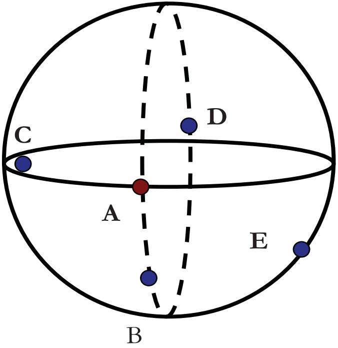
Distance



What is A closest to?

- No geometry: D
- With geometry: B

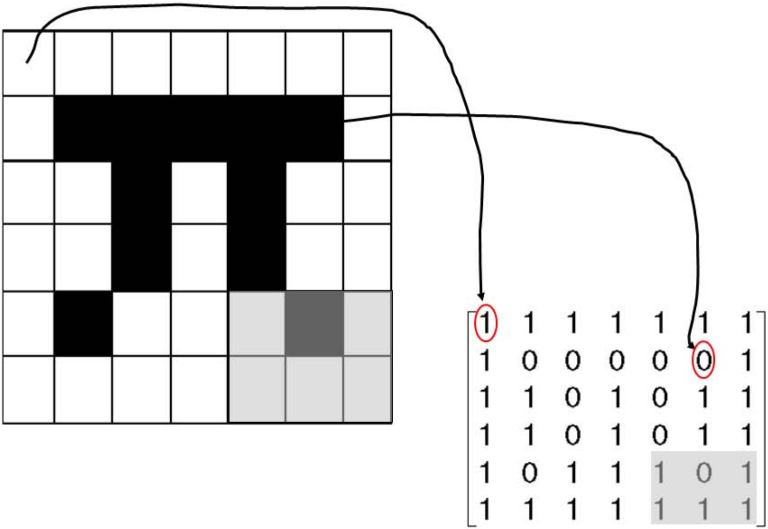
Distance



What is A closest to?

- No geometry: D
- With geometry: B

Data matrix



Data vector

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ \text{— — —} \\ 1 \\ 0 \\ 1 \\ 1 \\ 0 \\ 1 \\ \text{— — —} \\ \vdots \\ \text{— — —} \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

IMAGE $\longrightarrow$ MATRIX $\longrightarrow$ VECTOR

Approximation theorem

If we know the correct rank of A, e.g., by inspecting the singular values, then we can **remove the noise and compress the data** by approximating A by a matrix of the correct rank. One way to do this is to truncate the singular value expansion:

Theorem

If

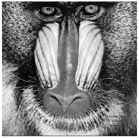
$$A_k = \sum_{i=1}^{k} \sigma_i u_i v_i^T \quad (1 \leq k \leq r)$$

then

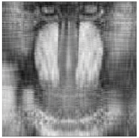
$$A_k = \min_{\mathrm{rank}(B)=k} \|A - B\|_2 \quad \textit{and} \quad A_k = \min_{\mathrm{rank}(B)=k} \|A - B\|_F .$$

An example

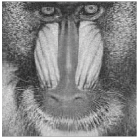
The error term of rank k approximation is given by the $(k+1)^{\text{th}}$ singular value σ_{k+1}.



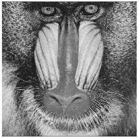
(a) full rank (rank 480)



(b) rank 10, rel. err. = 0.0551



(c) rank 50, rel. err. = 0.0305



(d) rank 170, rel. err. = 0.0126

General classification paradigm

1.5 **Preprocessing**
Geometric normalization,
Feature extraction, etc

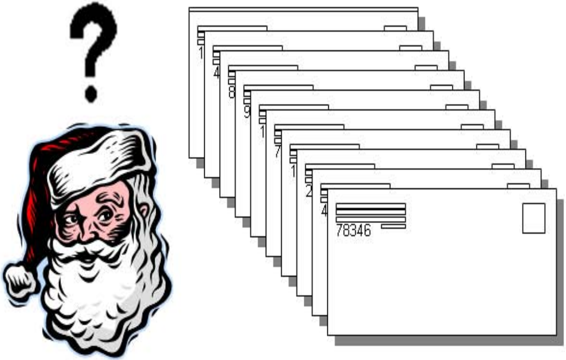
1. **Data collection**
Database creation
a.k.a. **Gallery**

2. **Present novel data**
a.k.a. **Probe**

3. **Classification**
Assign label to probe
and assess accuracy

Problem definition - globally

Santa thought to himself, “only if these mails can go to the right place according to their zip code”.



Handwritten digit classification

Problem. (Human) handwritten digits are sometimes very hard to recognize, even by human operators.

Importance. Accurate identification of the digits ensures a reliable delivery system.

Beneficiaries. Postal services (mail sorting), seaports (cargo registration), etc.

Even Santa Clause can benefit from an efficient digit classification algorithm.

Handwritten digit classification

Problem. (Human) handwritten digits are sometimes very hard to recognize, even by human operators.

Importance. Accurate identification of the digits ensures a reliable delivery system.

Beneficiaries. Postal services (mail sorting), seaports (cargo registration), etc.

Even Santa Clause can benefit from an efficient digit classification algorithm.

Handwritten digit classification

Problem. (Human) handwritten digits are sometimes very hard to recognize, even by human operators.

Importance. Accurate identification of the digits ensures a reliable delivery system.

Beneficiaries. Postal services (mail sorting), seaports (cargo registration), etc.

Even Santa Clause can benefit from an efficient digit classification algorithm.

Handwritten digit classification

Problem. (Human) handwritten digits are sometimes very hard to recognize, even by human operators.

Importance. Accurate identification of the digits ensures a reliable delivery system.

Beneficiaries. Postal services (mail sorting), seaports (cargo registration), etc.

Even Santa Clause can benefit from an efficient digit classification algorithm.

Handwritten digit classification

Problem. (Human) handwritten digits are sometimes very hard to recognize, even by human operators.

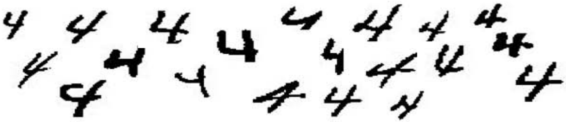
Importance. Accurate identification of the digits ensures a reliable delivery system.

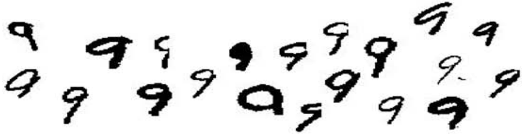
Beneficiaries. Postal services (mail sorting), seaports (cargo registration), etc.

Even Santa Clause can benefit from an efficient digit classification algorithm.

Problem definition - locally

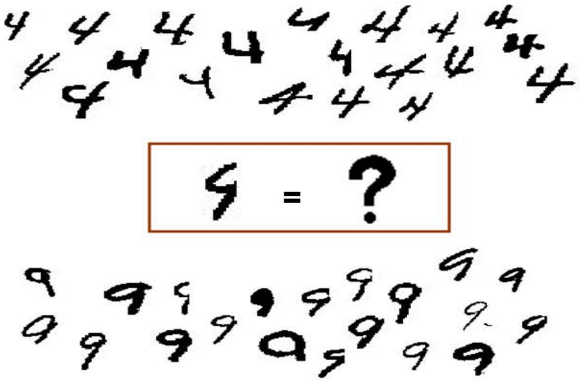
How do we tell a bunch of 4's from a bunch of 9's?





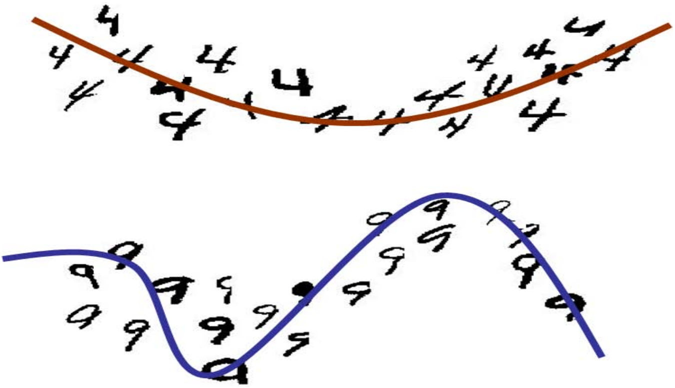
Problem definition - locally

Or, how do we tell whether a new digit is a 4 or a 9?



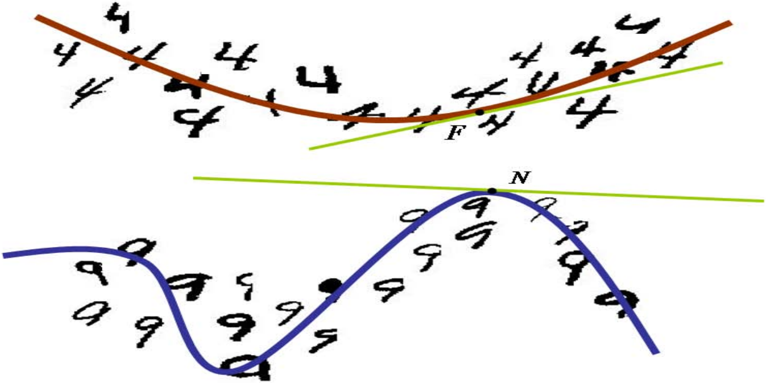
Digit manifolds

Imagine a high-D surface (red curve) where all 4's live on and a high-D surface (blue curve) where all 9's live on.



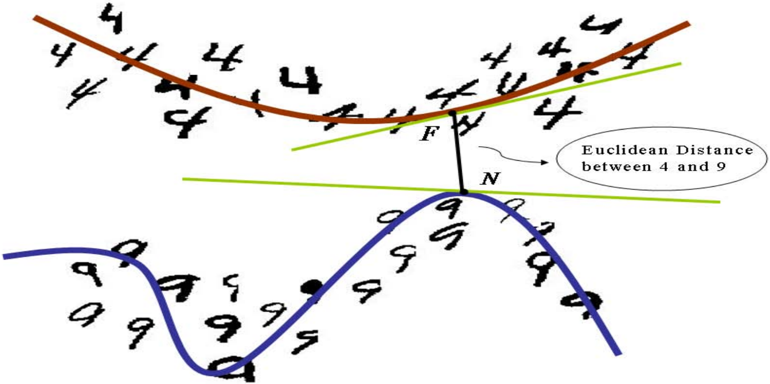
Tangent spaces - training

Create a Tangent Space of the 4's at F and create a Tangent Space of the 9's at N.



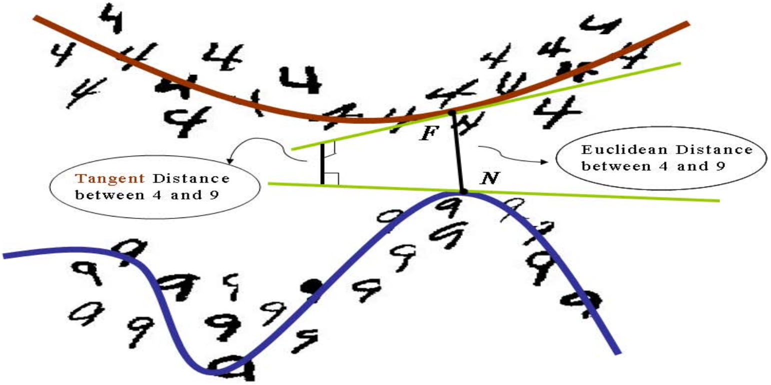
Dimensions of the tangent spaces depend on the degree of variations.

Euclidean distance



- Euclidean distance between each pair of 4 and 9 varies drastically.
- Calculation is time-consuming.

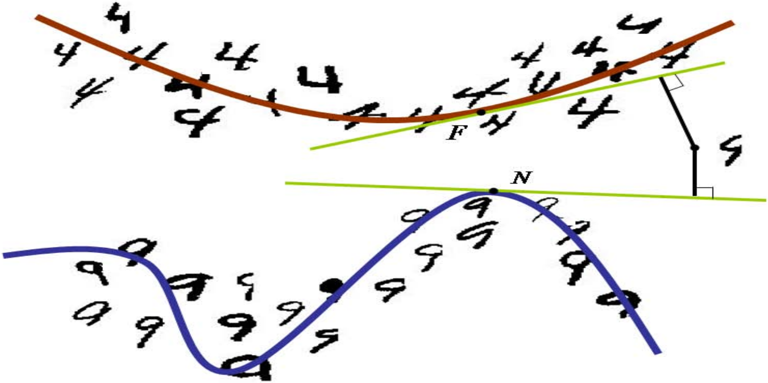
Tangent distance



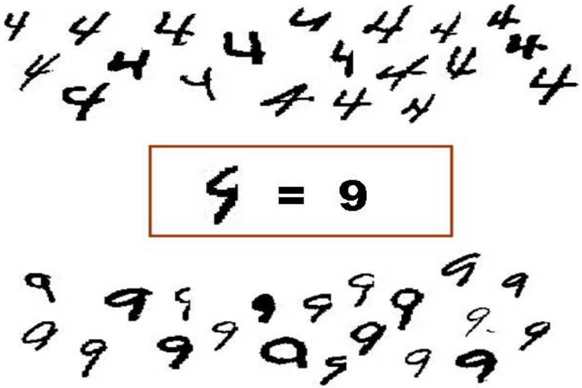
- Tangent distance captures the geometry.
- Calculation is efficient.

Classification

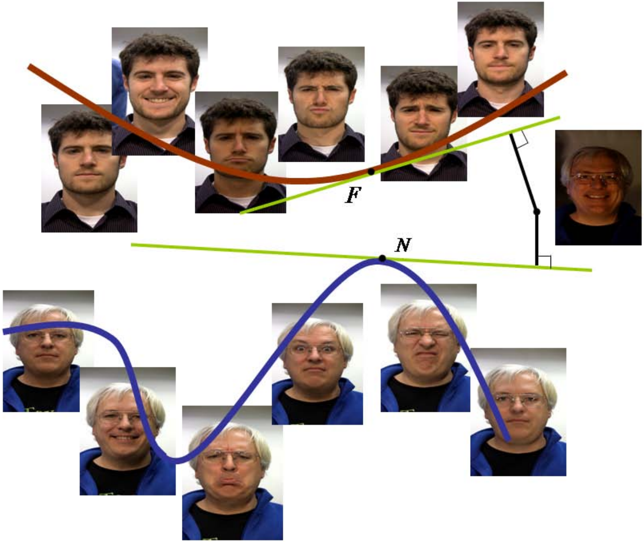
So, is it a 4 or a 9?



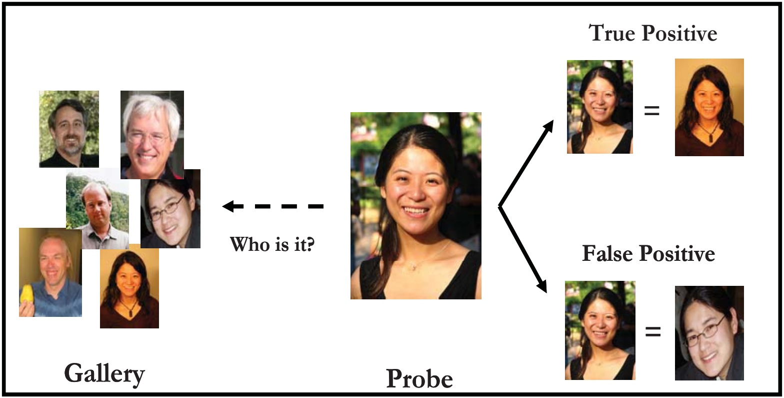
Classification result



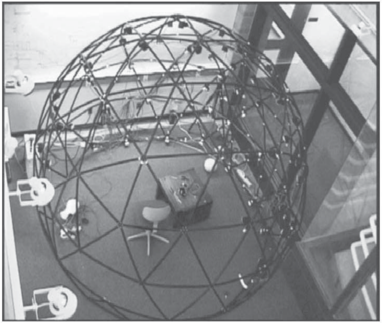
Face recognition



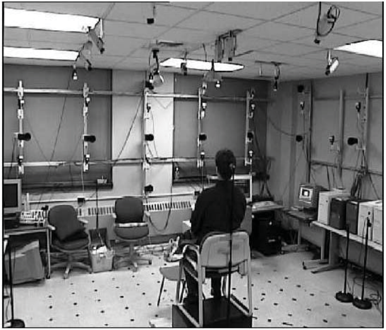
Face recognition paradigm



Illumination apparatus



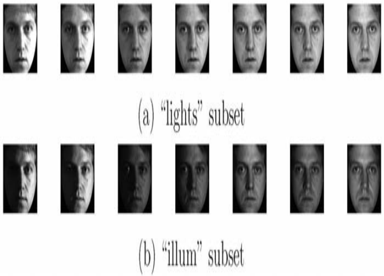
Yale Face Database B



CMU-PIE

Illumination images

Yale Face Database B



CMU-PIE

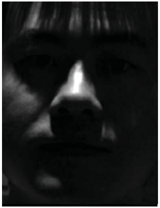
Empirical fact

Images of a single person seen under variations of illumination appear to be more difficult to recognize than images of different people [Zhao et al., 2003].

Subject 1

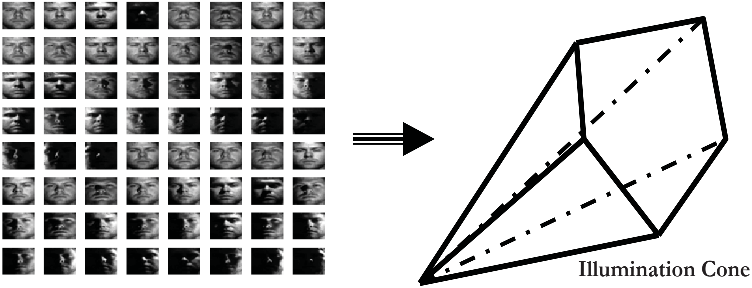
Subject 2

Can you tell who this is?



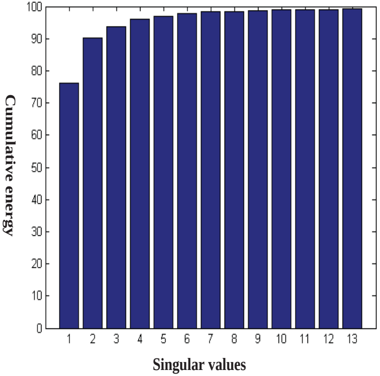
Geometric facts - 1

The set of m-pixel monochrome images of an object seen under general lighting conditions forms a convex polyhedral cone (illumination cone) in $\mathbb{R}^m$ [Belhumeur & Kriegman, 1998].

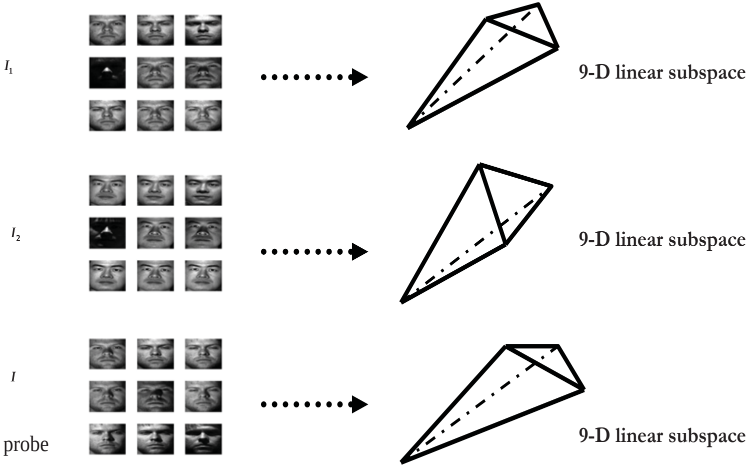


Geometric facts - 2

The illumination cone can be approximated by a 9-dimensional linear subspace [Basri & Jacobs, 2003], i.e., the illumination cone is low-dimensional and linear.

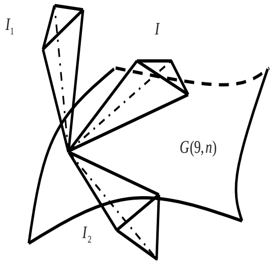


Set-up


I_1
9-D linear subspace
I_2
9-D linear subspace
I
probe
9-D linear subspace

Definition of $G(k, n)$

These illumination cones are all elements of a parameter space called the **Grassmannian (Grassmann manifold)**, $G(9, n)$, where n in the ambient dimension.



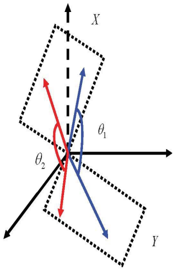
Definition

The *Grassmannian G(k,n)* or the *Grassmann manifold* is the set of k-dimensional subspaces in an n-dimensional vector space K^n for some field K, i.e.,

$$G(k, n) = \{W \subset K^n \mid \dim(W) = k\}.$$

Principal angles [Björck & Golub, 1973]

It turns out that any attempt to construct an unitarily invariant metric on $G(k, n)$ yields something that can be expressed in terms of the **principal angles** [Stewart & Sun, 1990].



Definition

If X and Y are two vector subspaces of $\mathbb{R}^m$, then the principal angles $\theta_k \in \left[0, \frac{\pi}{2}\right]$, $1 \leq k \leq q$ between X and Y are defined recursively by

$$\cos(\theta_k) = \max_{u \in X} \max_{v \in Y} u^T v = u_k^T v_k$$

subject to $\|u\| = \|v\| = 1$, $u^T u_i = 0$ and $v^T v_i = 0$ for $i = 1 : k-1$ and $q = \min\{\dim(X), \dim(Y)\} \geq 1$.

Grassmannian distances [Edelman et al., 1999]

These are the distance functions we will use to compare points on the Grassmann manifold.

Metric Name	Mathematical Expression
Fubini-Study	$d_{FS}(\mathcal{X},\mathcal{Y}) = \cos^{-1}\left(\prod_{i=1}^{k}\cos\theta_i\right)$
Chordal 2-norm	$d_{c2}(\mathcal{X},\mathcal{Y}) = \left\|2\sin\frac{1}{2}\theta\right\|_F$
Chordal F-norm	$d_{cF}(\mathcal{X},\mathcal{Y}) = \left\|2\sin\frac{1}{2}\theta\right\|_2$
Geodesic (Arc Length)	$d_g(\mathcal{X},\mathcal{Y}) = \|\theta\|_2$
Chordal (Projection F-norm)	$d_c(\mathcal{X},\mathcal{Y}) = \|\sin\theta\|_2$
Projection 2-norm	$d_{p2}(\mathcal{X},\mathcal{Y}) = \|\sin\theta\|_\infty$

Empirical result - database

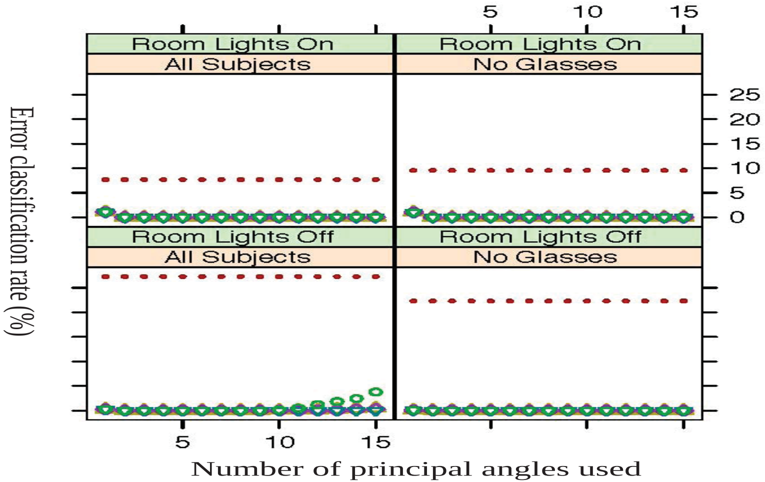
Since we are only concerned with the lighting variations, we fix the frontal pose, neutral expression and select the “illum” and “lights” subsets of CMU-PIE (68 subjects, 13 poses, 43 lightings, 4 expressions) [Sim et al., 2003] for experiments.

- lights: 21 illumination conditions with background lights **on.**
- illum: 21 illumination conditions with background lights **off.**

(a) “lights” subset

(b) “illum” subset

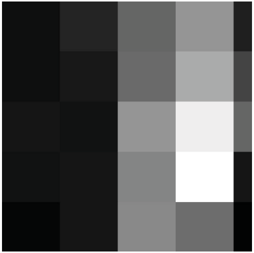
Empirical results



Robustness

If the data set is perfectly separable with the Grassmann method when using this kind of image [Chang et al., 2006a]:

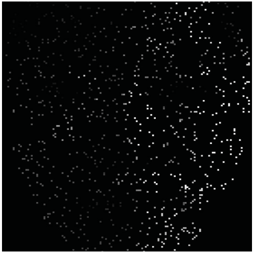
The data set is still perfectly separable with the Grassmann method when using this kind of image [Chang et al., 2007bc]:



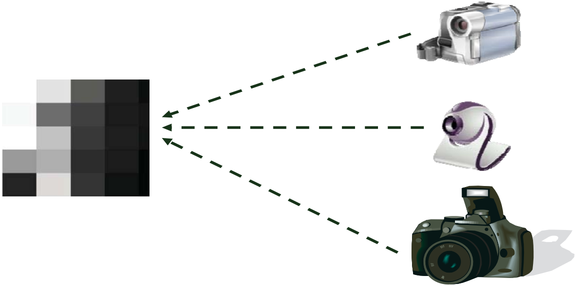
Robustness

If the data set is perfectly separable with the Grassmann method when using this kind of image [Chang et al., 2006a]:

The data set is still perfectly separable with the Grassmann method when using this kind of image [Chang et al., 2007bc]:

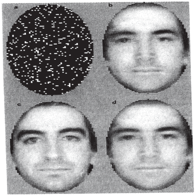


Potential use: low-res. illumination camera



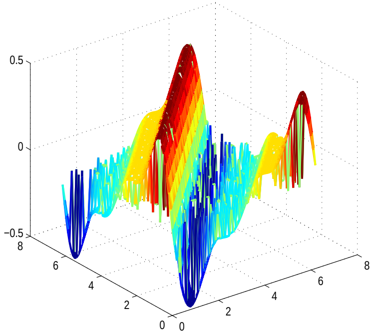
Large private databases of facial imagery can be stored at a resolution that is sufficiently low to prevent recognition by a human operator yet sufficiently high to enable machine recognition.

KL procedure for missing data

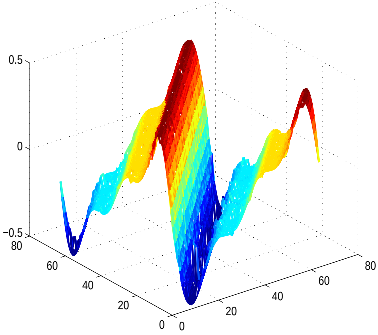


1. Initialize the missing data with the ensemble average.
2. Compute the first estimate of the KL basis.
3. Re-estimate the ensemble using the gappy approximation and the KL basis.
4. Re-compute the KL basis.
5. Repeat Steps 3–4 until stopping criterion is satisfied.

A gappy example

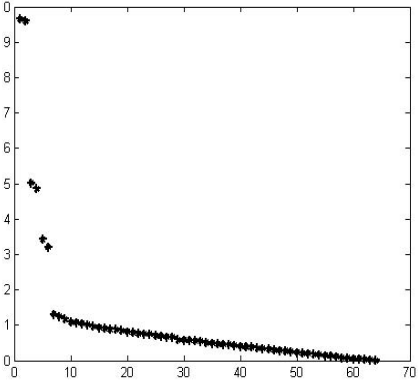


Gappy data

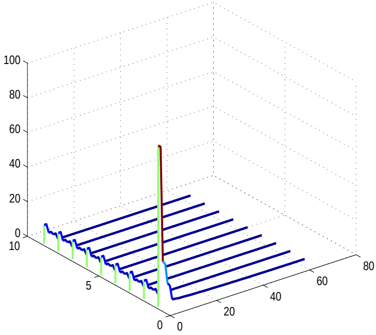


After 1 repair

Gappy example continued

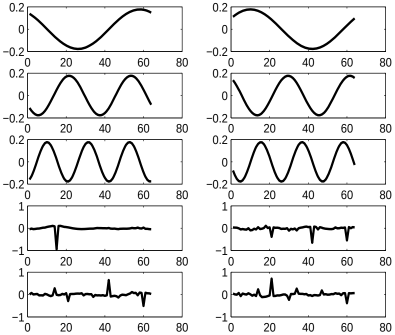


Singular values

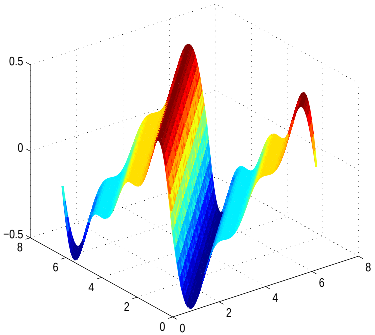


Eigenvalue vs. iteration

Gappy example continued



Eigenvectors of repaired data



Repaired

KL procedure for noisy data

- Decompose observed data into its *noise* and *signal* components:

$$\mathbf{x}^{(\mu)} = \mathbf{s}^{(\mu)} + \mathbf{n}^{(\mu)},$$

  or, in terms of data matrices,

$$X = S + N. \quad (S = \text{ signal}, N = \text{ noise})$$

- The optimal first basis vector, ϕ, is taken as a superposition of the data, i.e.,

$$\phi = \psi_1 \mathbf{x}^{(1)} + \cdots + \psi_P \mathbf{x}^{(P)} = X\psi.$$

- May decompose ϕ into signal and noise components

$$\phi = \phi_{\mathbf{n}} + \phi_{\mathbf{s}},$$

  where $\phi_{\mathbf{s}} = S\psi$ and $\phi_{\mathbf{n}} = N\psi$.

MNF continued

- The basis vector ϕ is said to have maximum noise fraction (MNF) if the ratio

$$D(\phi) = \frac{\phi_{\mathbf{n}}^T \phi_{\mathbf{n}}}{\phi^T \phi}$$

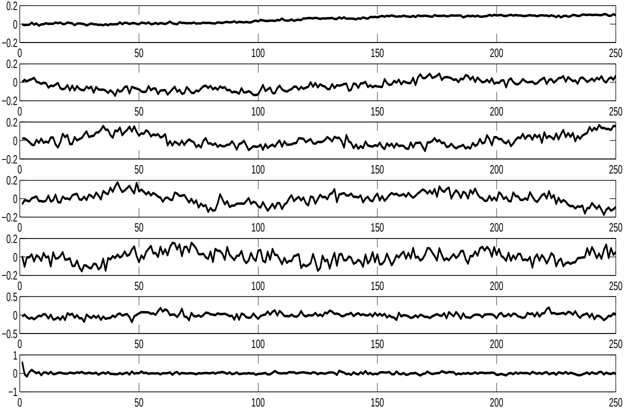
is a maximum.

- A steepest descent method yields the *symmetric definite generalized eigenproblem*

$$N^T N \psi = \mu^2 X^T X \psi.$$

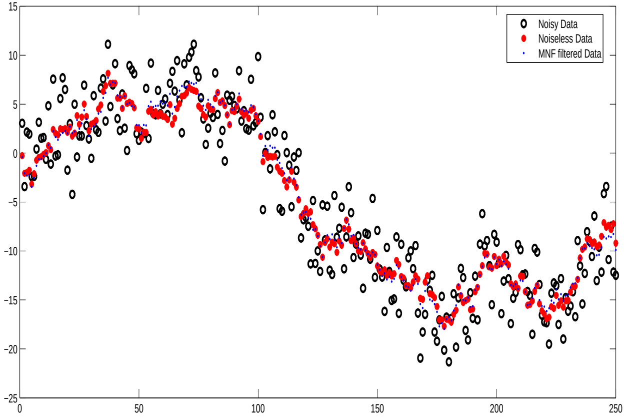
This problem may be solved without actually forming the product matrices $N^T N$ and $X^T X$, using the generalized SVD (gsvd).

A MNF example



6 observed noisy time series

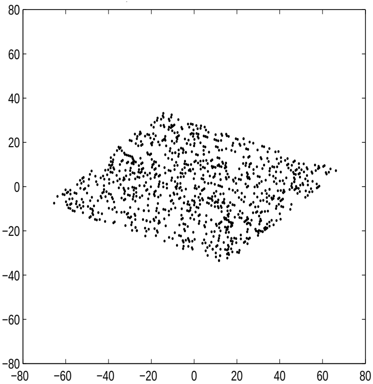
A MNF example



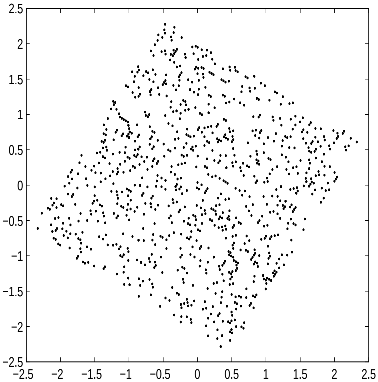
MNF filtered data

Whitening

Transform the observed data vectors **x**'s linearly so that the new vector **y**'s are uncorrelated and having variance one.



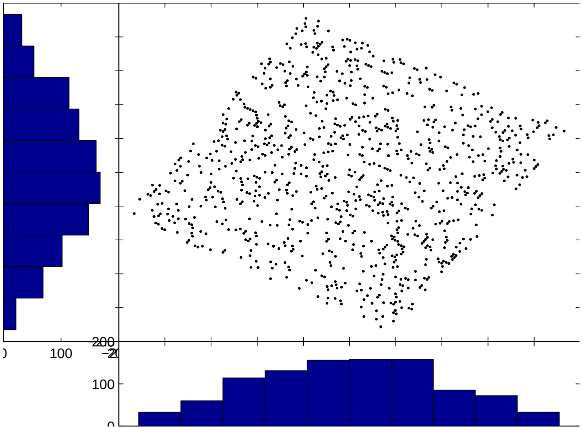
Observed data (**x**)



Whitened data (**y**)

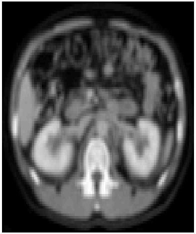
Resulting of Whitening

$$\mathbf{y} = \Lambda^{-1/2}\Phi^T\mathbf{x}, \text{ where } C = \Phi\Lambda\Phi^T \text{ and } C = \langle \mathbf{x}\mathbf{x}^T \rangle.$$

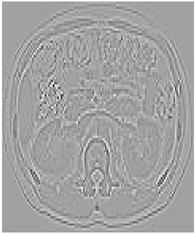


Convolution - sharpening

$$w(x,y) \star f(x,y) = \sum_{s=-a}^{a} \sum_{t=-b}^{b} w(s,t) f(x-s, y-t)$$
$$= \sum_{s=-a}^{a} \sum_{t=-b}^{b} f(s,t) w(x-s, y-t)$$



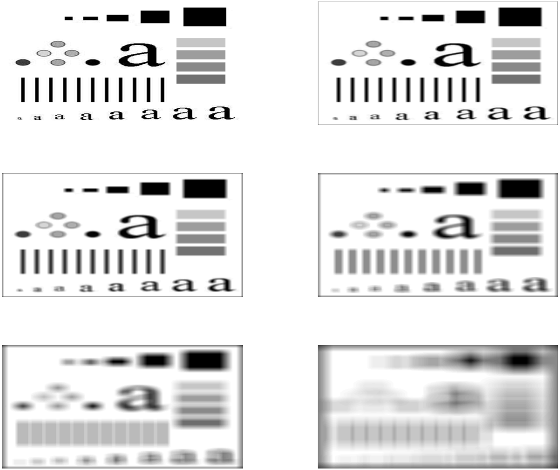
A blurred image



Laplacian edge filter

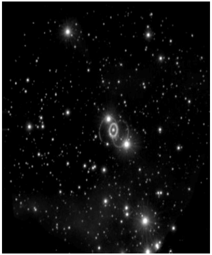
Enhanced image

Convolution - smoothing

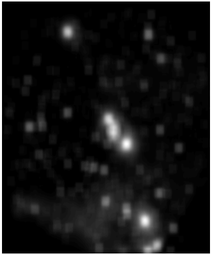


Convolution - threshold smoothing

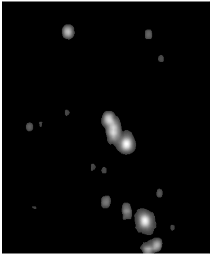
orginal



filtered with a 15 by 15 averaging filter

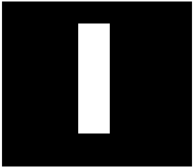


thresholded with 25% of highest intensity

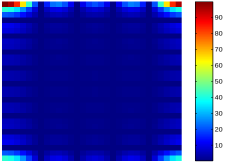


Fourier analysis

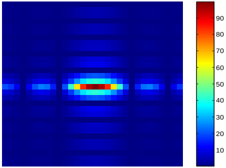
$$F(u,v) = \frac{1}{MN} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y) e^{-i2\pi\left(\frac{ux}{M}+\frac{vy}{N}\right)}$$



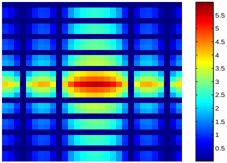
(a) Image.



(b) Spectrum.



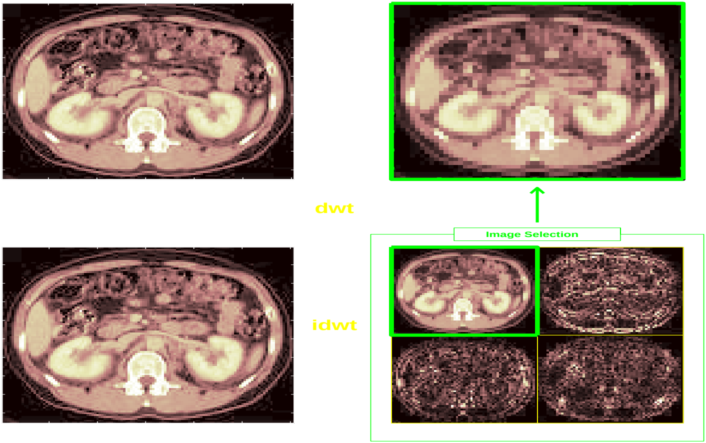
(c) Centered spectrum.



(d) log transform

Multiresolution analysis

$$X(b,a) = \frac{1}{\sqrt{|a|}} \int_{-\infty}^{\infty} x(t)\Psi^{*}\left(\frac{t-b}{a}\right)\, dt$$



References

[Basri & Jacobs, 2003] R. Basri & D. Jacobs, "Lambertian reflectance and linear subspaces", *PAMI, 25(2):218–233, 2003*.

[Belhumeur & Kriegman, 1998] P. Belhumeur & D. Kriegman, "What is the set of images of an object under all possible illumination conditions", *IJCV, 28(3):245–260, 1998*.

[Björck & Golub, 1973] A. Björck & G. Golub, "Numerical methods for computing angles between linear subspaces", *Mathematics of Computation, 27(123):579–594, 1973*.

[Chang et al., 2006a] J.-M. Chang, M. Kirby, H. Kley, J. R. Beveridge, C. Peterson, B. Draper, "Illumination face spaces are idiosyncratic", *Int'l Conf. on Image Proc. & Comp. Vision, 2: 390–396, 2006*.

[Chang et al., 2007b] J.-M. Chang, M. Kirby, H. Kley, J. R. Beveridge, C. Peterson, B. Draper, "Recognition of digital images of the human face at ultra low resolution via illumination spaces",*ACCV'07, LNCV, Springer, 4844: 733–743, 2007*.

References

[Chang et al., 2007c] J.-M. Chang, M. Kirby, C. Peterson, “Feature Patch Illumination spaces and Karcher compression for face recognition via Grassmannian”, *submitted to CVPR, 2007*.

[Edelman et al., 1999] A. Edelman, T. Arias, & S. Smith, “The Geometry of algorithms with orthogonality constraints”, *SIAM J. Matrix Anal. Appl., 20(2):303–353, 1999*.

[Stewart & Sun, 1990] G.W. Stewart & J.-G. Sun, “Matrix Perturbation Theory”, *Academic Press, 1990*.

[Zhao et al., 2003] W. Zhao, R. Chellappa, P. J. Phillips, A. Rosenfeld, “Face recognition: A literature survey”. *ACM Comp. Surv., 35(4):399–458, 2003*.