Section

1

Number Systems

Numbers and Their Computer Representation

Positional Notation

Decimal, Binary, Octal, and Hexadecimal, Nomenclature, Converting Between Number Systems, Binary Arithmetic (Subtraction with Comlements), Binary Codes (BCD, Error-Detection, Gray, and ASCII)

Introduction

Base 10 result of ten fingers

Arabic symbols 0-9, India created Zero and Positional Notation

Other Systems: Roman Numerals: essentially additive, Importance of Roman Numeral lies in whether a symbol precedes or follows another symbol. Ex. IV = 4 versus VI = 6. This was a very clumsy system for arithmetic operations.

Positional Notation (Positive Real Integers)

Fractional numbers will not be considered but it should be noted that the addition of said would be a simple and logical addition to the theory presented.

The value of each digit is determined by its position. Note pronunciation of 256 “Two Hundred and Fifty Six?

Ex. $256 = 2*10^2 + 5*10^1 + 6*10^0$

Generalization to any base or radix

Base or Radix = Number of different digit which can occur in each position in the number system.

$N = A_n r^n + A_{n-1} r^{n-1} + \ldots + A_1 r^1 + A_0 r^0$ (or simple $A_1 r + A_0$)

Introduction to Binary System

The operation of most digital devices is binary by nature, either they are on or off.

Examples: Switch, Relay, Tube, Transistor, and TTL IC

Thus it is only logical for a digital computer to in base 2.

Note: Future devices may not have this characteristic, and this is one of the reasons the basics and theory are important. For they add flexibility to the system.

In the Binary system there are only 2 states allowed; 0 and 1 (FALSE or TRUE, OFF or ON)

Example: Most Significant Bit
↓ High Order Bit

$1010 = 1*2^3 + 0*2^2 + 1*2^1 + 0*2^0 = 10_{10}$

↑ Least Significant Bit ↑ Denotes Base 10
Low Order Bit Usually implied by context

Bit = One Binary Digit (0 or 1)

This positional related equation also gives us a tool for converting from a given radix to base 10 - in this example Binary to Decimal.

Base Eight and Base Sixteen

Early in the development of the digital computer Von Neuman realized the usefulness of operating in intermediate base systems such as base 8 (or Octal)

By grouping 3 binary digits or bits one octal digit is formed. Note that $2^3 = 8$

Binary to Octal Conversion Table

$2^2 2^1 2^0$	
0 0 0	= 0
0 0 1	= 1
0 1 0	= 2
0 1 1	= 3
1 0 0	= 4
1 0 1	= 5
1 1 0	= 6
1 1 1	= 7

Symbols (not numbers) 8 and 9 are not used in octal.

Example: 100 001 010 110
4 1 2 $6_8 = 4*8^3 + 1*8^2 + 2*8^1 + 6*8^0 = 2134$

This is another effective way of going from base 2 to base 10

Summary: Base 8 allows you to work in the language of the computer without dealing with large numbers of ones and zeros. This is made possible through the simplicity of conversion from base 8 to base 2 and back again.

In microcomputers groupings of 4 bits (as opposed to 3 bits) or base 16 (2^4) is used. Originally pronounced Sexadecimal, base 16 was quickly renamed Hexadecimal (this really should be base 6).

Binary to Hex Conversion Table

$2^3 2^2 2^1 2^0$

0 0 0 0 = 0
0 0 0 1 = 1
0 0 1 0 = 2
0 0 1 1 = 3
0 1 0 0 = 4
0 1 0 1 = 5
0 1 1 0 = 6
0 1 1 1 = 7
1 0 0 0 = 8
1 0 0 1 = 9
1 0 1 0 = A
1 0 1 1 = B
1 1 0 0 = C
1 1 0 1 = D
1 1 1 0 = E
1 1 1 1 = F

In Hex Symbols for 10 to 15 are borrowed from the alphabet. This shows how relative numbers really are or in other words, they truly are just symbols.

Example: 1000 0101 0110
8 5 $6_{16} = 8*16^2 + 5*16^1 + 6*16^0 = 2134$

It is not as hard to work in base 16 as you might think, although it does take a little practice.

Conversion From Base 10 to a Given Radix (or Base)

Successive Division is best demonstrated by an example

```
2 | 43  \
2 | 21  \  1  Least Significant Bit
2 | 10  \  1
2 |  5  \  0
2 |  2  \  1
2 |  1  \  0
     0     1  Most Significant Bit
```

To get the digits in the right order let them fall to the right.

For this example: $43_{10} = 101011_2$ Quick Check (Octal) 101 011 = $5*8 + 3 = 43_{10}$

Another example: Convert 43_{10} from decimal to Octal

```
8 | 43  \
8 |  5  \  3
     0     5  Most Significant Bit
```

For this example: $43_{10} = 53_8$ Quick Check (Octal) $5*8 + 3 = 43_{10}$

Generalization of the procedure OR Why It Works

r	N	\		
r	N_1	\	A_0	Least Significant Bit
r	N_2	\	A_1	
r	N_3	\	A_2	
			A_3	
r	N_{n-1}	\		
r	N_n	\	A_{n-1}	
	0		A_n	Most Significant Bit

Where r = radix, N = number, A = remainder, and n = the number of digits in radix r for number N. Division is normally done in base 10.

Another way of expressing the above table is:

$N = r*N_1 + A_0$

$N_1 = r*N_2 + A_1$

$N_2 = r*N_3 + A_2$

:

$N_{n-1} = r*N_n + A_{n-1}$

$N_n = r*0 + A_n$

or (now for the slight of hand)

$N = r*(r*N_2 + A_1)+ A_0$ substitute N_1

$N = r^2N_2 + rA_1+ A_0$ multiply r through equation

$N = r^2(r*N_3 + A_2) + rA_1+ A_0$ substitute N_2

:

$N = A_nr^n + A_{n-1}r^{n-1} + \ldots + A_1r^1 + A_0r^0$ $\therefore$

Nomenclature

Bit = 1 binary digit

Byte = 8 bits

Nibble = one half byte = 4 bits

Word = Computer Dependent

Binary Arithmetic

Binary Addition

Binary addition is performed similar to decimal addition using the following binary addition rules:

0 + 0 = 0

0 + 1 = 1

1 + 0 = 1

1 + 1 = 10 (0 with a carry of 1)

Examples:

Problem ☞	$21_{10} + 10_{10} = 31_{10}$	$45_{10} + 54_{10} = 99_{10}$	$3_{10} + 7_{10} = 10_{10}$
	10101_2	101101_2	011_2
	$+\ 01010_2$	$+\ 110110_2$	$+\ 111_2$
	11111_2	1100011_2	1010_2
Check ☞			$1*2^3 + 0*2^2 + 1*2^1 + 0*2^0$
			$1*8 + 0*4 + 1*2 + 0*1 = 10_{10}$

Octal Addition

Octal addition is also performed similar to decimal addition except that each digit has a range of 0 to 7 instead of 0 to 9.

Problem ☞	$21_{10} + 10_{10} = 31_{10}$	$45_{10} + 54_{10} = 99_{10}$	$3_{10} + 7_{10} = 10_{10}$
	25_8	55_8	3_8
	$+\ 12_8$	$+\ 66_8$	$+\ 7_8$
	37_8	143_8	12_8
Check ☞	$3*8^1 + 7*8^0$	$1*8^2 + 4*8^1 + 3*8^0$	$1*8^1 + 2*8^0$
	$3*8 + 7*1 = 31_{10}$	$64 + 32 + 3 = 99_{10}$	$8 + 2 = 10_{10}$

Hexadecimal Addition

Hex addition is also performed similar to decimal addition except that each digit has a range of 0 to 15 instead of 0 to 9.

Problem ☞	$21_{10} + 10_{10} = 31_{10}$	$45_{10} + 54_{10} = 99_{10}$	$3_{10} + 7_{10} = 10_{10}$
	15_{16}	$2D_{16}$	3_{16}
	$+\ 0A_{16}$	$+\ 36_{16}$	$+\ 7_{16}$
	$1F_{16}$	63_{16}	A_{16} (not 10)
Check ☞	$1*16^1 + 15*16^0$	$6*16^1 + 3*16^0$	$10*16^0$
	$16 + 15 = 31_{10}$	$96 + 3 = 99_{10}$	10_{10}

Binary Multiplication

Decimal	**Binary**
11_{10}	1011_2
x 13_{10}	x 1101_2
33_{10}	1011_2
11_{10}	0000_2
	1011_2
	1011_2
143_{10}	10001111_2

Check ☞ $8*16^1 + 15*16^0$

$128 + 15 = 143_{10}$

Binary Division

Decimal	**Binary**
21_{10}	10101_2
5_{10}) 105_{10}	101_2) 1101001_2
10	101
05	110
05	101
00	101
	101
	000

Check ☞ $1*16^1 + 5*16^0$

$16 + 5 = 21_{10}$

Practice arithmetic operations by making problems up and then checking your answers by converting them back to base 10 via different bases (i.e., 2, 8, and 16).

How a computer performs arithmetic operations is a much more involved subject and has not been dealt with in this section.

Complements and Negative Numbers OR Adding a Sign Bit

Addition, Multiplication, and Division is nice but what about subtraction and negative numbers? From grade school you have learned that subtraction is simply the addition of a negative number. Mathematicians along with engineers have exploited this principle along with modulo arithmetic — a natural outgrowth of adders of finite width — to allow computers to operate on negative numbers without adding any new hardware elements to the arithmetic logic unit (ALU).

Sign Magnitude

Here is a simple solution, just add a sign bit. To implement this solution in hardware you will need to create a subtractor; which means more money.

		sign	magnitude
Example:	- 2 =	1	0010_2

Ones Complement

Here is a solution that is a little more complex. Add the sign bit and invert each bit making up the magnitude — simply change the 1's to 0's and the 0's to 1's.

		sign	magnitude
Example:	- 2 =	1	1101_2

To subtract in 1's complement you simply add the sign and magnitude bits letting the last carry bit (from the sign) fall into the **bit bucket**, and then add 1 to the answer. Once again let the last carry bit fall into the bit bucket. The bit bucket is possible due to the physical size of the adder.

	0 1010_2	10
+	1 1101_2	+(-2)
	0 1000_2	8
+	1_2	Adjustment
	0 1001_2	

Although you can now use your hardware adder to subtract numbers, you now need to add 1 to the answer. This again means adding hardware. Compounding this problem, ones complement allows two numbers to equal 0 (**schizophrenic zero**).

Twos Complement

Here is a solution that is a little more complex to set up, but needs no adjustments at the end of the addition. There are two ways to take the twos complement of a number.

Method 1 = Take the 1's complement and add 1

	$0\ 0010_2$	2 ← start
+	$1\ 1101_2$	1's complement (i.e. invert)
+	1_2	add 1
	$1\ 1110_2$	

Method 2 = Move from right to left until a 1 is encountered then invert.

$0\ 0010_2$	start ☞ 2_{10}
0_2	no change
10_2	no change but one is encountered
110_2	invert ☞ change 0 to 1
1110_2	invert ☞ change 0 to 1
$1\ 1110_2$	invert ☞ change 0 to 1

Subtraction in twos complement is the same as addition. No adjustment is needed, and twos complement has **no schizophrenic zero** although it does have an additional negative number (see How It Works).

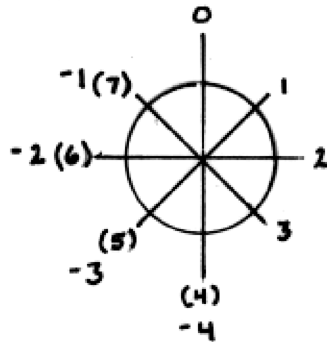
	$0\ 1010_2$	10
+	$1\ 1110_2$	+(-2)
	$0\ 1001_2$	8

Examples:

Problem ☞	$33_{10} - 19_{10} = 14_{10}$	$69_{10} - 84_{10} = -15_{10}$
	$0\ 100001_2$	$0\ 1000101_2$
	$+\ 1\ 101101_2$	$+\ 1\ 0101100_2$
	$0\ 001110_2$	$1\ 1110001_2$
Check ☞	convert to intermediate base $E_{16} = 14_{10}$	convert back to sign magnitude $-\ 0001111_2$ convert to intermediate base (16) $-\ F_{16} = -\ 15_{10}$

Why It Works

Real adders have a finite number of bits, which leads naturally to modulo arithmetic — the bit bucket.



Overflow

With arithmetic now reduced to going around in circles, positive numbers can add up to negative and vice-versa. Two tests provide a quick check on whether or not an "Overflow" condition exists.

Test 1 = If the two numbers are negative and the answer is positive, an overflow has occurred.

Test 2 = If the two number are positive and the answer is negative, an overflow has occurred.

> If computers were calculators and the world was a perfect place, we would be done. But they are not and so we continue by looking at a few real world problems and their solutions.

Character Codes OR Non-Numeric Information

Decimal Number Problem

Represent a Decimal Numbers in a Binary Computer. A binary representation of a decimal number, a few years ago, might have been "hard wired" into the arithmetic logic unit (ALU) of the computer. Today it, more likely than not, is simply representing some information that is naturally represented in base 10, for example your student ID.

Solution

In this problem, ten different digits need to be represented. Using 4 bits 2^4 or 16 combinations can be created. Using 3 bits 2^3 or 8 combinations can be created. Thus 4 bits will be required to represent one Decimal Digit. It should here be pointed out how 16 combinations can be created from 4 bits (0000 - 1111) while the largest numeric value that can be represented is 15. The reason that the highest numeric value and the number of

combinations are different, is due to zero (0) being one of the combinations. This difference points up the need to always keep track of wetter or not you are working zero or one relative and what exactly you are after — a binary number or combinations.

The most common way of representing a decimal number is named Binary Coded Decimal (BCD). Here each binary number corresponds to its decimal equivalent, with numbers larger than 9 simply not allowed. BCD is also known as an 8-4-2-1 code since each number represents the respective weights of the binary digits. In contrast the Excess-3 code is an unweighted code used in earlier computers. Its code assignment comes from the corresponding BCD code plus 3. The Excess-3 code had the advantage that by complementing each digit of the binary code representation of a decimal digit (1's complement), the 9's complement of that digit would be formed. The following table lists each decimal digit and its BCD and Excess-3 code equivalent representation. I have also included the negative equivalent of each decimal digit encoded using the Excess-3 code. For instance, the complement of 0100 (1 decimal) is 1011, which is 8 decimal. You can find more decimal codes on page 18 of "Digital Design" by M. Morris Mano (course text).

Binary Coded Decimal (BCD)		**Excess-3**		
Decimal Digit	Binary Code 8-4-2-1	Decimal Digit	Binary Code	9's Compliment
0	0000	N/A	0000	1111
1	0001	N/A	0001	1110
2	0010	N/A	0010	1101
3	0011	0	0011	1100
4	0100	1	0100	1011
5	0101	2	0101	1010
6	0110	3	0110	1001
7	0111	4	0111	1000
8	1000	5	1000	0111
9	1001	6	1001	0110
N/A	1010	7	1010	0101
N/A	1011	8	1011	0100
N/A	1100	9	1100	0011
N/A	1101	N/A	1101	0010
N/A	1110	N/A	1110	0001
N/A	1111	N/A	1111	0000

Alphanumeric Character Problem

Represent Alphanumeric data (lower and upper case letters of the alphabet (a-z, A-Z), digital numbers (0-9), and special symbols (carriage return, line feed, period, etc.).

Solution

To represent the upper and lower case letters of the alphabet, plus ten numbers, you need at least 62 (2x26+10) unique combinations. Although a code using only six binary digits

providing 2^6 or 64 unique combinations would work, only 2 combinations would be left for special symbols. On the other hand a code using 7 bits provides 2^7 or 128 combinations, which provides more than enough room for the alphabet, numbers, and special symbols. So who decides which binary combinations correspond to what character. Here there is no “best way.” About thirty years ago IBM came out with a new series of computers which used 8 bits to store one character (2^8 = 256 combinations), and devised the Extended Binary-Coded Decimal Interchange Code (EBCDIC pronounced ep-su-dec) for this purpose. Since IBM had a near monopoly on the computer field, at that time, the other computer makers refused to adopt EBCDIC, and that is how the 7bit American Standard Code for Information Interchange (ASCII) came into existence. ASCII has now been adopted by virtually all micro-computer and mini-computer manufacturers. The table below shows a partial list of the ASCII code. Page 23 of the text lists all 128 codes with explanations of the control characters.

DEC	HEX	CHAR	DEC	HEX	CHAR
32	20		64	40	@
33	21	!	65	41	A
34	22	“	66	42	B
35	23	#	67	43	C
36	24	$	68	44	D
37	25	%	69	45	E
38	26	&	70	46	F
39	27	‘	71	47	G
40	28	(	72	48	H
41	29	)	73	49	I
42	2A	*	74	4A	J
43	2B	+	75	4B	K
44	2C	,	76	4C	L
45	2D	-	77	4D	M
46	2E	*	78	4E	N
47	2F	/	79	4F	O
48	30	0	80	50	P
49	31	1	81	51	Q
50	32	2	82	52	R
51	33	3	83	53	S
52	34	4	84	54	T
53	35	5	85	55	U
54	36	6	86	56	V
55	37	7	87	57	W
56	38	8	88	58	X
57	39	9	89	59	Y
58	3A	:	90	5A	Z
59	3B	;	91	5B	[
60	3C	<	92	5C	\
61	3D	=	93	5D	]
62	3E	>	94	5E	^
63	3F	?	95	5F	_

The word “string” is commonly used to describe a sequence of characters stored via their numeric codes — like ASCII).

Although ASCII requires only 7 bits, the standard in computers is to use 8 bits, where the leftmost bit is set to 0. This allows you to code another 128 characters (including such things as Greek letters), giving you an *extended character set*, simply by letting the leftmost bit be a 1. This can also lead to a computer version of the tower of Babel. Alternatively, the leftmost bit can be used for detecting errors when transmitting characters over a telephone line. Which brings us to our next problem.

Synthesis

Although ASCII solves the communication problem between English speaking computers, what about Japanese, Chinese, or Russian computers which have different, and in all these examples, larger alphabets?

Communication Problem

Binary information may be transmitted serially (one bit at a time) through some form of communication medium such as a telephone line or a radio wave. Any external noise introduced into the medium can change bit values from 1 to 0 or visa versa.

Solution

The simplest and most common solution to the communication problem involves adding a *parity bit* to the information being sent. The function of the parity bit is to make the total number of 1's being sent either odd (odd parity) or even (even parity). Thus, if any odd number of 1's were sent but an even number of 1's received, you know an error has occurred. The table below illustrates the appropriate parity bit (odd and even) that would be appended to a 4-bit chunk of data.

Synthesis

What happens if two binary digits change bit values? Can a system be devised to not only detect errors but to identify and correct the bit(s) that have changed? One of the most common error-correcting codes was developed by R.W. Hamming. His solution, known as a Hamming code, can be found in a very diverse set of places from a Random Access Memory (RAM) circuit to a Spacecraft telecommunications link. For more of error correcting codes read pages 299 to 302 of the text.

Although detecting errors is nice, preventing them from occurring is even better. Which of course brings us to our next problem.

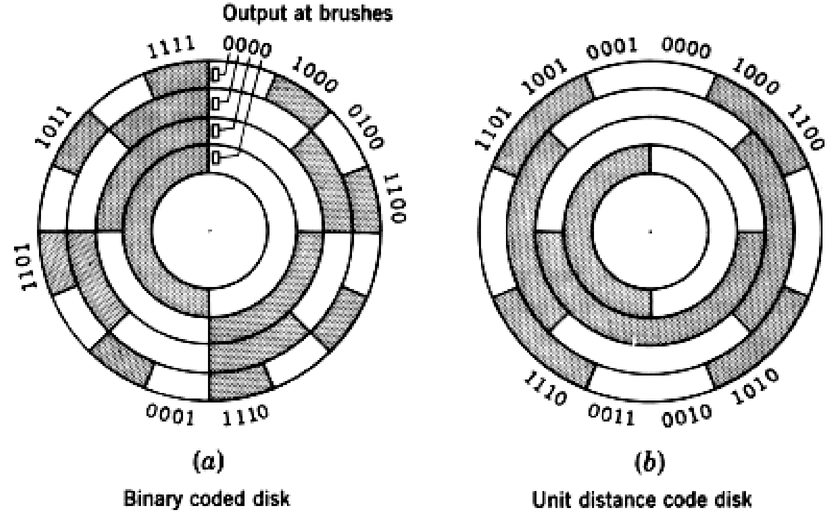
Shaft Encoder Problem

As a shaft turns, you need to convert its radial position into a binary coded digital number.

Solution

The type of coder which will be briefly described below converts a shaft position to a binary-coded digital number. A number of different types of devices will perform this conversion; the type described is representative of the devices now in use, and it should be realized that more complicated coders may yield additional accuracy. Also, it is generally possible to convert a physical position into an electric analog-type signal and then convert this signal to a digital system. In general, though, more direct and accurate coders can be constructed by eliminating the intermediate step of converting a physical position to

an analog electric signal. The Figure below illustrates a coded-segment disk which is coupled to the shaft.



(a) Binary coded disk (b) Unit distance code disk

The shaft encoder can be physically realized using electro-mechanical (brush) or electro-optical technology. Assuming an electro-optical solution, the coder disk is constructed with bands divided into transparent segments (the shaded areas) and opaque segments (the unshaded areas). A light source is put on one side of the disk, and a set of four photoelectric cells on the other side, arranged so that one cell is behind each band of the coder disk. If a transparent segment is between the light source and a light-sensitive cell, a 1 output will result; and if an opaque area is in front of the photoelectric cell, there will be a O output.

There is one basic difficulty with the coder illustrated: if the disk is in a position where the output number is changing from 011 to 100, or in any position where several bits are changing value, the output signal may become ambiguous. As with any physically realized device, no matter how carefully it is made, the coder will have erroneous outputs in several positions. If this occurs when 011 is changing to 100, several errors are possible; the value may be read as 111 or 000, either of which is a value with considerable errors. To circumvent this difficulty, engineers use a "Gray," or "unit distance," code to form the coder disk (see previous Figure). In this code, 2 bits never change value in successive coded binary numbers. Using a Gray coded disk, a 6 may be read as 7, or a 4 as 5, but larger errors will not be made. The Table below shows a listing of a 4-bit Gray code.

Decimal	Gray Code
0	0000
1	0001
2	0011
3	0010
4	0110
5	0111
6	0101
7	0100
8	1100
9	1101
10	1111

11	1110
12	1010
13	1011
14	1001
15	1000

Synthesis

Gray code is used in a multitude of application other than shaft encoders. For example, CMOS circuits draw the most current when they are switching. If a large number of circuits switch at the same time unwelcome phenomena such as “Ground Bounce” and “EMI Noise” can result. If the transistors are switching due to some sequential phenomena (like counting), then these unwelcome visitors can be minimized by replacing a weighted binary code by a Gray code.

If the inputs to a binary machine are from an encoder using a Gray code, each word must be converted to conventional binary or binary-coded decimal bit equivalent. How can this be done? Before you can answer this question, you will need to learn about Boolean Algebra — what a coincidence, that’s the topic of the next Section.