Hi, this is Professor Jen-Mei Chang at California State University, Long Beach. In this video lesson today, we'll look at how to use transformations to graph rational functions. A rational function of the form $r$ of $x$ equals $ax$ plus $b$ over $cx$ plus $d$ can be graphed by shifting, stretching, and/or reflecting the graph of $f$ of $x$ equals $1$ over $x$. So notice that the rational function has to be of the form of a linear polynomial over a linear polynomial in order for us to do this. So it's essential basically to look at the graph of $f$ of $x$ equals $1$ over $x$ and then just apply the known transformations to it to get to the resulting $r$ of $x$. For example, if I want to sketch $r$ of $x$ equals $2$ over $x$ minus $3$, we first let $f$ of $x$ be the original $1$ over $x$ function. Then we realize that $r$ of $x$ can be obtained by applying $2$ transformations to $f$ of $x$.

Namely, I can first do a horizontal shift to the right $3$ units to obtain $f$ of $x$ minus $3$. Then to get the $2$ on top, I multiply the entire function by $2$. Which is equivalent to the effect of stretching vertically by a factor of $2$. We label $f$ of $x$ $2f$ of $x$ minus $3$, step $1$. And then $f$ of $x$ minus $3$ to $2$ times $f$ of $x$ minus $3$, step $2$. Then graphically, this corresponds to going from the original $1$ over $x$ graph, which is in the first and the third quadrant. Looking like this with the horizontal asymptote at the $x$ axis and the vertical asymptote on the $y$ axis. Then step $1$ corresponds to having the vertical asymptote at $x$ equals $3$. Having the vertical asymptote at $x$ equals $3$ and then just shifting everything accordingly. Then in step $2$, we're going to stretch vertically by a factor of $2$. Which means that for every $x$ value, we're going to double its $y$ value.

So for example, if this was $x$ equals $4$, then its $y$ value is at $1$. We're going to double that to $2$. For $x$ equals $5$, its $y$ value is $1/2$, so we doubled it. It's going to be $1/4$. And you can kind of see it basically gets -- come a little bit more extreme in $y$ direction. The basic shape is still the same. So this would become the final graph of the $r$ of $x$ function. Let's look at another example next. Here, I want to sketch the function $s$ of $x$ equals $3x$ plus $5$ over $x$ plus $2$. And I want to contrast this example with the previous one. We notice that the polynomial function on top has the same degree as the polynomial function on the bottom. They're both degree $1$. Versus in the previous case, the polynomial function on top is a constant function, which has $1$ degree less than the polynomial function on the bottom.

Now, in this case here in example $1$, we can simply just apply the transformations and do it right away. But in the second example here, I actually have to perform a long division first to bring down the power in the remainder before I can actually apply the transformations. With the long division on the side, I have $3x$ plus $5$ as the dividend. The divisor is $x$ plus $2$. I'm going to get $3$ as my first term of the quotients. That's $3x$ plus $6$ and then subtract those $2$. I get $5$ minus $6$, and that is negative $1$. Which means by the long division process, I can rewrite the original functions $3x$ plus $5$ over $x$ plus $2$ as the quotient $3$. Plus remainder over the divisor, which is negative $1$ over $x$ plus $2$. Now, if we let $f(x)$ be $1$ over $x$, then I can obtain the $s$ of $x$ by a series of $3$ steps. First, I can make $f$ of $x$ plus $2$, which corresponds to a horizontal shift to the left $2$ units.
Next, I will get negative f of x plus 2 by applying that negative sign in front of the entire function. Which has the effect of reflecting around the x axis. And finally, I can add 3 to the entire expression. So that's going to be 3 minus f of x plus 2. And adding the 3 has the effect of translating the entire function up by 3 units. And you can verify that. If I were to apply these operations to the original function 1 over x, you should get back to the s of x as desired. Because this is telling you to do 3 minus f of x plus 2. Well, that is to replace the entire function input x by this new input of x plus 2. This gives me 1 over x plus 2. Which is what we have for s of x. So that means these transformations must be correct. To sketch, we started with the original function f of x. Step 1 tells us to shift to the left by 2 units.

Now the vertical asymptote becomes x minus 2. So your graph should be near around x minus 2. In step 2, we needed it to reflect around the x axis. So take all y values. Basically just change it to its opposite y values. The first portion, which sits above the x axis, now becomes below the x axis. And then the second portion over here, which originally sits below the x axis, now is going to sit above the x axis. Finally, in steps 3, we're going to apply the vertical shift up by 3 units. Originally, we have the horizontal asymptote on the x axis. Now, the horizontal asymptote is going to be on the line y equals 3. We get a picture that looks like this. And that's our final graph of the function s of x.