Hello. Welcome to Linear Algebra on Demand brought to you by Professor Jen-Mei Chang at California State University Long Beach. In this video lesson today we'll look at the definition of inner product and the length in high dimensional spaces. We need a notion to study the quantity AX hat minus B where X hat is the best approximation to the linear system AX equals B. In order to speak of AX hat minus B, we have to speak of the length of AX hat minus B. Therefore, that gives us the motivation to study inner product and the length. We start with a definition of inner product. You might be familiar with inner product defined in three-dimensional space, which is called a dot product. It's defined as such. If I tell you a vector U has three components -- U1, U2, and U3 -- vector V has also three components -- V1, V2, and V3.

Then the dot product of U and V is defined to be U1, V1 plus U2, V2, plus U3, V3. In a summation notation, we can write this as the summation of UI, VI where I goes from 1 to 3. If you want to transfer this into a matrix notation, we can write this as the vector U transpose times V. You'll remember we typically realize vector as a column vector. That is why when you have the first vector, just a row vector do we use the U transpose notation. So we can realize this as the real vector. The second vector, then, we can realize as a column vector, V. Then by doing a row column multiplication in the vector, vector multiplication, we get exactly the definition as above. [Inaudible] and the generalization of the dot product in the high dimensional space is defined precisely as the following. The inner product or the dot product between a n-dimensional vector, U, and n-dimensional vector, V, is precisely U transpose V. Or we could write it as V transpose U.

The inner product is commutative in this sentence. And in summation notation, this is precisely UI times VI where I goes from 1 to n, n being the dimension of the vector. We can expand this definition to the definition of length easily because speaking of the length of a vector simply use definition of the inner product as such. You might have learned this from a calculus course. The length of a vector with n entries is defined to be square root of each entry squared in sum up. But that is also to say we’re doing the inner product or the dot product of a vector with itself. You can kind of think of it as a vector V, V1 through Vn, row-column multiplication, like that. You get V1 squared plus V2 squared and all the way to Vn squared, just like that. Of course, you need a square root with it.

So essentially when we speak of the length of the vector, we're saying it's the square root of its own inner product. Or, in other words, the length of a vector squared is precisely its own inner product, which is V transpose V. We often think of this equation more than this equation. If, for example, the length of the vector is equal to 1, then we say that the vector is of unit length. Then we can always normalize a vector by dividing its length so that the resulting vector is of unit length. And that's always possible. So the process of making a vector into a unit vector is called normalization. So when we speak of normalizing a vector, we're essentially speaking of the process of making it into a unit length. So, for example, if I ask you to normalize the following vector, then essentially what you're doing is you're looking for a unit vector that is in the same direction defined by the vector V.

To do that we first find the length of the vector V. That is to do V transpose V and then take the square root. So we're going to square each component and then add it all together. That's 1
plus 4 plus 4 plus 0. And the resulting number, we're going to take the square root of that. And this is equal to 3. In order to find a vector $U$ that is of the same direction as $V$, it needs to be, of course, the same direction of either for we need that original $V$, but we want it to be of unit length. Therefore, we're going to divide out its length. This new vector is going to be the same direction as before, which is $V$, and of unit length.

So that is going to be $1 \over 3$ times the original $V$. Notice that we used this transpose notation because we've written the vector as a row vector. With the transpose it turns the vector into a column vector. So essentially this becomes one-third, negative two-thirds, two-thirds, and 0. Of course, you can easily check that this new vector is of unit length. You can always believe that if you've done your calculation right, this will always have to be of unit length. And the reason is because if you then take the length of this new vector, then what you get is going to be the length of $V$ on top divided by the length of the length $V$. Well, the length $V$ here is already a real number, so then the length of that is still going to be itself, which is always equal to 1. You can be sure that this process will always produce a unit vector. Let's look at one more example.

In this example, I want to find a unit vector that is a basis for a subspace $W$ in $\mathbb{R}^2$ that is spanned by the vector $X$ in the direction two-thirds $1$. Well, pictorially what this is saying that you've got your space of $\mathbb{R}^2$ here and you've got the subspace that is spanned by the direction $X$. Well, $X$ is two-thirds $1$, so it's about right here. This is the subspace by $X$. You're looking for a basis for that subspace, so essentially you're looking for a direction that has a unit length. Which then you take the span of that gives you the entire subspace of $W$. In order to get this basis that is of a unit vector, let's call this $Y$, we basically repeat the same process as before. We're going to define $Y$ to be the same direction as the $X$, but then of unit length. Well, the length of $X$ here is two-thirds squared plus 1 squared and, oh, under square roots. And this calculation then gives us a square root of 13 divided by 3.

Therefore, my new vector $Y$ has to be $1 \over 3$ times the X vector. After some simplification I get the vector of $2 \over \sqrt{13}$ and $3 \over \sqrt{13}$. Of course, this will be a perfectly good answer. If you then take the opposite direction of it, which is just a negative of this vector, it will also be a correct answer, as well. The question just says find a unit vector. There are actually two choices. Now that we have a notion of inner product and norm, we can speak of distances between points in high dimensional space. In 1D, if you have just 2 points on a real number line, then what is the distance between them? We simply find the distance between $A$ and $B$ to be the absolute value of $A$ minus $B$.

This gives us the amount of difference of those two things. In 2D, for example, if I want to measure the difference, or the distance between two vectors, say vector $U$ and vector $V$, what I really want to capture is the difference between their tips, the blue vector. And using the law of parallelogram, we quickly notice that this is the vector $U$ minus $V$. If we just want to capture the amount of differences without any direction, I would then take the norm of this new vector, so we say the distance between the vector $U$ and vector $V$ to be the norm of $U$ minus $V$. So for vectors in high dimensional spaces, the distance is defined exactly the same way. The distance between vector $U$ and vector $V$ both in $\mathbb{R}^n$, is precisely the norm of their difference.
So, for example, if I gave you the vector $U$, that is of 1, 2, 3, and the vector $V$ is of 3, 4, 5, then the distance between $U$ and $V$ is precisely the norm of the vector that is $U$ minus $V$. And do this component-wise subtractions gives us negative 2, negative 2, and negative 2. And then you take the norm of this vector, which is essentially taking the square root of the inner product with itself. Well, that's going to be 2 squared plus 2 squared plus 2 squared, and then all under square roots. This, then, gives us the result of square root of 12. Alright, so this concludes a quick discussion on inner product, norm, and distances in high dimensional space of $\mathbb{R}^n$. 
