Hello. Welcome to Linear Algebra on Demand brought to you by Professor Jen-Mei Chang at California State University Long Beach. In this media lesson today we'll look at the diagonalization of matrices as a similarity transformation and also the preservation of eigenvalue under such similarity transformation. We know that a matrix A is diagonalizable if there exists a diagonal matrix D and invertible matrix P, such that I can write a matrix A as a PDP inverse. If this is the case, we actually call the matrix A similar to the matrix D, and this is why. The definition of the similar matrices is such. If matrix A is similar to B and denoted by the similarity sign, if there exists an invertible matrix P such that I can write the matrix A as PBP inverse. And this is why we call the matrix A similar to the matrix D here. The D matrix right here plays the role of the B matrix here in the definition of similarity transformation. And because of this definition, it shouldn't be hard to see that if I state A is similar to matrix B, then B must be similar to the matrix A, as well, simply by letting a new matrix Q be the inverse of P.

Then I can write the matrix B as P inverse A times P. I simply do that by multiplying the P inverse on both sides to get rid of this P and I would multiply the P matrix on the right-hand side of this equation to get rid of this P inverse. But then that becomes a P on the right-hand side of the matrix A. But because Q is P inverse, I can then write this as QAQ inverse. And that -- now it shouldn't be hard to see that by this definition B is similar to the matrix A where Q is invertible matrix. Now if you start with the matrix A, you can obtain this new matrix B that is written as P inverse AP. Then the transformation that takes the original matrix A to this new matrix of P inverse AP is called a similarity transformation. It is actually interesting to know that the eigenvalues of the matrix A are preserved under the similarity transformation. And in other words, the eigenvalues of the matrix A is exactly the same set of the eigenvalues of this new matrix P inverse A times P.

We'll look at why in the proof next. If we were to phrase what we just said in the theorem, then the [inaudible] response to saying that if the n by n matrices are similar, then they have the same characteristic polynomial and, hence, the same eigenvalues. And here's the proof. If the matrix A is similar to matrix B, that means there must exist an invertible matrix P such that I can write A as PBP inverse. I can also re-write that equation into equivalent following with P inverse A -- I can also re-write this equation into its equivalent form of B equals P inverse A times P. The claim is that the matrix A and the matrix B share the same characteristic polynomial and, therefore, the same eigenvalues. We need to look at P minus lambda I in the determinants of that quantity.

But that is saying to look at the determinants of P inverse AP minus lambda I, the right-hand side. Well, the identity matrix can be rewritten as P inverse times P. And I like to do that because I like to factor out the P inverse on the left-hand side, and I like to factor out the P on the right-hand side. So I can rewrite this expression here as the following: P inverse on the left and P on the right. In the middle, I have the matrix A and lambda I. Well, I has been split into the left and the right, so there's just a lambda. But remember looking at subtract a matrix with a scaler I need to have an ending matrix associated with it. This becomes lambda I. You can verify this must be true by distributing it back. If you were to distribute this back, you get P inverse times A, and that quantity multiplied by the P should give you the first term. And if you distribute this second term, you get P inverse times I and then times P.
Well, that's exactly just $D\lambda$. And then we're going to use the property of determinants. It says that the determinant of a product is equal to the product of the determinants. I can then split this entire determinant quantity into three terms. And since the determinants are just a bunch of real numbers, I can then reorder them however I want. In particular I'm going to do the multiplication of determinant $P$ with the $P$ inverse first. The reason why is then because I can combine these two things using kind of the inverse process and make it into the determinant of the product $P$ inverse times $P$. But that is saying it's the determinant of the identity matrix times the determinant of $A$ minus $\lambda I$. We also know the determinant of identity matrix is just equal to 1.

So this boils down to the determinant $A$ minus $\lambda I$. But basically what this proof has shown that the characteristic polynomial of the matrix $B$ is exactly the same as the characteristic polynomial of the matrix $A$. Hence, we've shown that they share the same characteristic polynomial and, of course, the eigenvalues are just the zeroes of this polynomial. When you set both equal to zero, you should get exactly the same roots to that polynomial. I want to make a quick note here to warn people that doing a row operation on the matrices typically changes the eigenvalues of that matrix. On the other hand, the similarity transformation does not change the eigenvalues. You should be able to come up with a quick example to illustrate this. For example, if I let the matrix $A$ be the 2 by 2 with the 1 and 2 on the diagonal and a 0, 4 on the off-diagonal, and simply doing a half of row 2 turns this matrix into a 1, 4 and 0, 1. The eigenvalue of this original matrix $A$ is 1 and 2, but the eigenvalue of this row equivalent matrix $B$ is 1 and 1. So doing a row operation on the matrix definitely will change the eigenvalues.