Welcome to Algebra On demand, brought to you by Professor Jen-Mei Chang at California State University, Long Beach. In this video lesson today we'll look at the geometric and algebraic definition of Eigen value and Eigen vectors and also the definition of Eigen space, all on a 2 by 2 example. So start with the definition of an Eigen vector and Eigen value. An Eigen vector of a num [Assumed Spelling] matrix A is a nonzero vector x, such that A, x is equal to lambda x for some scaler lambda. This lambda is called an Eigen value with the associated Eigen vector x. And I want to emphasize a few keywords here. Eigen stands for characteristics, and notice that it has to be for a square and bio matrix in order for an Eigen vector and an Eigen value pair to exist. In an Eigen vector an Eigen value pair must satisfy the equation A, x equals lambda x, and the Eigen vector must be nonzero or nontrivial.

Geometrically what this means is you think of the matrix A as a transformation that somehow transform the vector x into different vector. In this equation here says if you start with a direction x here, and after the matrix transformation it's going to do something to it, either stretch it out or rotate it or refract it somehow. Eventually this new vector that you obtained after the transformation is some scale of multiple of the original vector that you started with, okay? And that is why we use Eigen because we're looking for the settable axis, such that it's getting fixed after the transformation. So it's more of an intrinsic characteristic of the matrix A. For every matrix A you expect to have a set of Eigen vectors and Eigen value pair that corresponds to that matrix. So this is more of an intrinsic characteristic for that matrix. So a quick example of Eigen value, Eigen vector pair, if you have a matrix A, 3, 1, negative 2, 0, then the vector U here which is negative 1, 1 of this calculation we'll see that 8 times U is equal to negative 5 and negative 1, which is definitely not any constant multiple of the original U you started with.

There's no lambda you can combine with, that you can multiply this original vector U to get to the negative 5, negative 1. And, therefore, so this implies that U is not an Eigen vector of A, which means that this direction of U, negative 1, 1 cannot be fixed after the action of A. There's no way that you can bring this vector back to its original direction after the affect of transformation A. On the other hand, if we looked at another vector V, that is 2, 1, then we would get 4, 2 after we multiplied A times V. We notice that this is two times the original vector V. We were able to come out of the equation A, V equals some scale multiple of the original V. This 2 right here would be called the Eigen value, and the V here would be called the Eigen vector, and we would call the pair 2 and V the Eigen value, Eigen vector pair for the matrix A. And geometrically this is saying that the direction 2, 1 can be fixed after the affect of transformation A. So somehow after you hit this vector V with the transformation A, it comes back to twice as long as the original vector.

Now that we understand kind of what the Eigen value and Eigen vector means, let's look at how we actually find it next. The Eigen value, Eigen vector must satisfy the equation A, x equals lambda x, so we need to manipulate this matrix into something that we can actually work with. We reword this matrix here, first, by subtracting lambda x, both sides. And then I'm going to factor out the x, so this becomes A minus lambda, and the x on the right-hand side, however, I can't really do the subtractions between a matrix with the scaler, so I'm going to do this with the identity matrix to make it into a diagonal matrix with the lambdas on the diagonal. And this is equivalent to the previous equation. And then I look at this equation here and ask
ourselves when do we have a nontrivial or non-zero solution to the homogeneous system? Because, remember, that's the definition of the Eigen vector, it has to be a non-zero vector that satisfies this equation.

And homogeneous system, A minus lambda I, x equals zero, has nontrivial solution whenever the matrix A minus lambda I is non-invertible. In the matrix A minus lambda I is not invertible, which it normally is, the determinant of A minus lambda I is equal to zero. In this equation right here, here's all the characters to the equations. Next, I'm going to illustrate how to use the characters to the equation to find Eigen value and Eigen vector pair for a matrix, for a 2 by 2 matrix A. For example, the matrix A is 3, 1, negative 2, 0. The characteristic equation, which will denote by fee [Assumed Spelling] of lambda, is a function of lambda, is equal to the determinant of A minus lambda I.

We use the vertical bar for determinants of the matrix. The original matrix A is 3, negative 2, 1, 0. I'm going to subtract lambda I, but remember lambda I corresponds to this matrix with the lambda on the diagonal everywhere else. So essentially it's only diagonal entries that's going to have the affect when we do the subtraction. So that gives us 3 minus lambda, and 0 minus lambda. The determinant of the 2 by 2 matrix is easily done, simply by doing the product of the diagonal entries, minus the product of the diagonal entries. So we get 3 minus lambda, times minus lambda, minus negative 2 times 1. All together we have num square minus 3 lambda and plus 2.

This is a quadratic equation with one unknown. We can easily solve this using either factoring technique or a quadratic formula. Simply factor this, we get lambda minus 1 and lambda minus 2. And remember that we're looking for the solution such that the determinant is equal to zero. Before starting this equation I get lambda equals 1 or lambda equals 2. What this says is that the matrix A has two Eigen values, one is equal to 1 and the other one is equal to 2. I'm going to find the corresponding Eigen vectors associated with the Eigen value separately. For the first Eigen value, lambda equals 1, let's find its Eigen vector.

The associated Eigen vector must [inaudible] the equation A, x equals lambda x, and that means that I have to write A, x equals one times x. And so on this I will get A, x minus I, x, equals 0, which then allows me to write A minus I, times x, equal to 0. I could have gotten here from the very beginning simply by looking at this equation, the [inaudible] equation A minus lambda I. Lambda here is equal to 1, so essentially you are solving for A minus 1 I, x equals 0. So I want to make a note here, I'm looking for the Eigen vector, this is essentially solving for the homogeneous system A minus lambda I, quantity x, equals zero for the corresponding associated Eigen value lambda. So by doing that I will get the system 2, 1, negative 2, negative 1, augment that with the 0 vector. We'll actually simply solve this system quickly. The last equation here gives us x, 2 equals x, 1. The equation here is x, 1 minus x, 2, equals 0. So when I solve for the solution x, which is x, 1, x, 2, I would get x, 2 and factor out the 1, 1. The space that contains only Eigen vectors, associated with these Eigen, with this Eigen value is denoted by the E sub lambda.

I'm going to actually make this a lambda so 1, because there are two different Eigen values here. The first one denoted by lambda 1. So the Eigen space for the Eigen value, lambda 1,
this is the Eigen space, the lambda 1. This is exactly the [inaudible] space of A minus lambda 1, times I. Or we actually have found all its members. Its member is a set of all things of the form, some constant multiple times the direction 1, 1. This 1, 1 is precisely the basis for the Eigen space E sub lambda 1 because everything else is just a constant multiple of it. And in this case geometrically the Eigen space for lambda 1 corresponds to a line in R 2 that passes through the origin, having the direction 1, 1.

Now, similarly, you can do exactly this for the second Eigen value, lambda 2, which is equal to 2. The Eigen space for lambda 2 is precisely the num space for A minus lambda 2 times I, and that is equal to A minus 2 I. In solving this homogeneous system gives us the relationship x, 1 minus 2, x, 2, equals 0, and x, 1 equals 2, x, 2. And this then tells us that the Eigen space for lambda 2 is a set of all things of the form, some constant multiple of 2 and 1. And, again, the geometric interpretation of the Eigen space for lambda 2 corresponds to the line in R, 2 that passes through the points 2, 1 and 0, 0. This concludes the quick discussion of the definition and the computation of Eigen value and Eigen vector pair.