Next, we'll describe the spanning set theorem, which gives a way to start with a big set of things that span a subspace and then gradually reduce it down to a basis by removing linearly dependent ones. Let's start with some notations here. Let S be the set that contains elements v₁, v₂, through vp, which is a subset of the vector space V. And let H be the span of these elements. So, to get a mental picture of what these things mean. Think of the vector space V as this huge [inaudible] world. And the set S is a collection that has just these p points in them. OK. So, you can just grab those vp points. Together they form the set S. And then H is the span of these p points. [Inaudible] significantly more points than the set S because the span is the set of all linear combinations.

So, you expect the span to be something bigger than this, but potentially less than the entire set of vector space V. Now the spanning set theorem says if it happens that one of these guys in the collection is a linear combination of the other ones, then the set in S by throwing that point away will still form a basis for the set H. So, what that means is if, for example, some Vⱼ is linear combination of the other ones in this collection, S, then the set S by throwing that point away will still span the set H. Then also if this set H is not empty, then we know for sure that some subset in there, some subset of the collection S will have to be a basis for H. So, essentially the spanning set theorem gives us a recipe of finding a basis. So, the idea is that you start with some bigger collection of points. And then if the span of these points coincides with the span of a smaller collection of points in there, then by throwing that extra points away, you're not losing anything.

So, I'm going to illustrate this idea in this following example. All right. Given three points, v₁, v₂, and v₃. And we're going to let the set H be the span of this collection of points, v₁, v₂, v₃. So, S is the collection that condense v₁, v₂, v₃. And the H is the span of all of that. We want to show that the span of v₁, v₂, and v₃, which essentially is H, is actually the same sets as a span of simply using v₁ and v₂. And if the span of v₁ and v₂ gives rise to the exact same set as the span of v₁, v₂, v₃, then by throwing away the point in v₃, you are not really losing any points by doing that. And once we can show that, and then no more points in the set here is linearly dependent of each other, then this will automatically be a basis for the set H. OK. So, the idea is that you start with something bigger.

So, the set here S has got three elements in there. But then if you can show that one of the elements is a linear combination of the other ones, then by throwing that away to form a smaller set, you're not losing anything. Therefore your basis will be embedded in this collection. So, then once we can show this, then we can say that the basis is the v₁, v₂. Let's call this set here G. In order to show set equality, I will need to show set inclusions both ways. That is to show that everything that is found in H can also be found in G and vice versa. Everything that's found in G can be found in H. So, if I pick an arbitrary element in H, I should be able to show that the same elements can be written as linear combinations in G. So, since x is in H that means x must be a linear combination of v₁, v₂, and v₃. With some coefficient C i's. So, assuming that these C i's are actually known. My goal is then to write x in terms of linear combinations of v₁ and v₂ only. So, if I can find a way to rewrite v₃ as a linear combination of v₁ and v₂, then I should be there.
OK. SO, in order to write \( v_3 \) as a combination of \( v_1 \) and \( v_2 \), I notice first that the third component in \( v_3 \) is negative 5. And because the third component in \( v_2 \) is 0, it doesn't really matter what constant multiply, put right in front of it, you'll never get out of 0. So, it needs to all come from the \( v_1 \). So, since I need a negative 5, I'm just going to multiply \( v_1 \) by 5. Similarly, for the first component here, because it's already a 0 in \( v_1 \), I can never get anything out of that, I need to manipulate the two in order to get to the 6th. So, I'm going to do 3 times \( v_2 \). And add those two together. The only thing I need to check is the middle component actually matches up. I need to get to 16 by doing this operation. So, if I do 5 times 2, that's 10. And 3 times 2 that would be 6. So, yes, I do get the 16. So, this is the way to write a linear combination with these three. So, since this is the case, that means that I can now rewrite the \( x \) with this new expression of \( v_3 \) in place.

So, everything else is the same. Now with the \( v_3 \) I get 5\( v_1 \) plus 3\( v_2 \). Group according to the \( v_i \). I get C1 plus 5C3 times the \( v_1 \). For the \( v_2 \) I get C2 plus 3C3 times \( v_2 \). So, [inaudible] just another number. Like this is just some other number. So, I would successfully write the element \( x \) in terms of linear combinations of \( v_1 \) and \( v_2 \) only. Therefore this implies the \( x \) must be in an element of \( G \). And remember, \( x \) was arbitrary. So, this can be done for any point in \( H \). Therefore, \( H \) must be a subset of \( G \). So, we've done inclusion one way. We need to show inclusion the other way. So, now I need to pick an arbitrary point in \( G \) and show that it can be found in \( H \). So, let a \( y \) be some arbitrary point in \( G \). So, that means I can write \( y \) as a linear combination of \( v_1 \) and \( v_2 \). If that is the case, then certainly \( y \) is a linear combination of \( v_1 \), \( v_2 \), and \( v_3 \). So, I can simply just use 0 for the \( v_3 \).

So, this should be a big obvious thing done. Right away \( y \) is an element in \( H \). So, we get inclusion the other way. So, \( G \) is a subset of \( H \). So, once you've shown that left hand side is a subset of the right hand side, and the right hand side is a subset of the left hand side, then these two sets must be equal. So, this tells me that \( H \) must be equal to \( G \), which means that the collection of points that are in \( G \) and in \( H \) are identical. Essentially what this result has shown you is that you just need the two elements, \( v_1 \) and \( v_2 \) to span the whole thing. So, this spanning set theorem says if you can span the same set using fewer points, then the set of fewer points would then be a basis for the original set. So, this result immediately tells us that the collection of \( v_1 \), \( v_2 \) forms a basis for the set \( H \). Of course you have to verify that \( v_1 \) and \( v_2 \) actually are linearly independent, and that shouldn't be hard to see because the position of where 0 shows up. They have to be linearly independent. OK. So, this concludes the illustration of the spanning set theorem.