Hi, this is Professor Jen-Mei Chang at California State University Long Beach. In this video lesson today we're going to learn about the concept that is a very important idea in [inaudible] algebra and also everywhere else. And this idea of spaces is essentially a collection of vectors that span a vector space or subspace in the most efficient way. We'll [inaudible] the idea with an example that we're very familiar with already.

So, consider the space of R3. You have learned in multivariable calculus there is a set of vectors that you use to represent every other vectors in the space. And essentially they are i, j, and k. So, the vector i is denoted by 1, 0, 0. That is the vector of unit lengths pointing in the direction of x1. Vector j is 0, 1, 0. Unit length vector that's pointing in the direction of x2. In the vector k, that is 0, 0, 1 pointing in the direction of x3. And before in calculus you would represent any general vector, for example, vector 2, 5, negative 4 using linear combinations of i, j, and k essentially because the vector 2, 5, negative 4 is 2 times vector 1, 0, 0, plus 5 times 0, 1, 0 plus negative 4 times 0, 0, 1. This is 2i plus 5j minus 4k. So, the vectors i, j, and k here serves as a collection of vectors that you can use to write every other vectors as a linear combination of them.

And then why do choose i, j, and k? Why not any other collection of three vectors? Well, there's something special about this collection of i, j, and k that they're orthogonal to each other meaning that they form a 90-degree angle with each other. And also that they are of unit length. You should be able to write this vector u as a linear combination of i, j, and k. And this is true for any u, any u you pick in this space. So, that makes i, j, and k very special. So, then the question becomes can you do this with a fewer set of vectors? So, can you write u as a linear combination of just two vectors in this space? So, for example, just use i and j. Can you get to u using simply i and j? Well, hopefully you know that the span of i and j, so span is a set of all linear combinations of i and j, will only give you the space in the x1, x2 plane. So, you'll never be able to get anything outside of that.

So, u seems like it looks outside of the x1, x2 plane, you would never be able to represent that using linear combinations of i and j. And similarly, if you choose any other two vectors, say j and k, then you would be able to write just linear combinations of those two points would just be x2, x3 plane. So, it seems that simply by using two vectors in this space you cannot write any other vector as a linear combination of them. So, you need the three. So, you need at least three vectors to write every vector in the space. So, that's a very important fact. And that kind of is related to why it's called R3. So essentially you need a minimum of three vectors to write all vectors as a linear combination.

Then the question becomes do you need four vectors? Do you need five vectors? Do you need more than three to write all of them? So, if I tell you I want to write u as a linear combination of the i, j, and k and also something else. Say, let's say 1, 2, 3. OK? The question is, is it really necessary to have that fourth vector? Well, it turns out that it's not because notice that 1, 2, 3 is already a linear combination of i, j, and k with the coefficient 1, 2, and 3. All right. So, this is already a linearly dependent set. So, if you actually put, you set i, j, k with this 1, 2, 3, and that already forms a linearly dependent set. So, you really don't need this vector there. So, there are really two important concepts going on here is that, one, in order to represent a general vector in R3, you need at least three vectors to do it.
So, that's a minimum of three in order to span the space of R3. And you actually need at most three to do that because any time you have more than three vectors then you can write the other ones as linear combinations of the one before already. So, you really don't need more than three. So, this is at least three and also at most three. And that's why it makes this number so special. And the set of three vectors that you can use to span a space and also form a linearly independent sets would be called a basis for the space. So, essentially a basis for the space is a set that is, that spans the space minimally. Minimally meaning that it has the minimal amount of vectors that does it. And then also linearly independent maximally. It's a set of vectors that are linearly independent. And if you add any more to it, it's not going to become linearly independent anymore.

So, those two conditions combined together makes up the definition of the basis. If we have a set h that is a subspace of a vector space v. I'll use shorthand notation v dot s for vector space. Then an index just means that there is a way that you can label the elements in there. I am going to call this index set a script B and in the way we're labeling them it's just using b1, b2, all the way to bp. So, there are p of them. We call that the set here B is a basis for the subset H if two conditions hold. One is that the sets B has to form a linearly independent set. And two, if I take the span of these b i's, I need to get the entire H.

So, that just means that I can write any vectors in linear combination of these b i's. And not only that, this is d, kind of the most compact way of writing it. If you add any more then it's going to become a linearly dependent set, and that's not the most efficient way of writing the linear combination. So, a key remark is that in order to find a basis for a set or to determine whether a set is a basis for a subspace or a vector space, we need to check whether or not the set is a minimal spanning set as well as a maximal linearly independent set. Let me give you some quick examples of a basis and non-basis for R3. Let me tell you actually how to create a basis that is not a trivial one with just i, j, k. OK. So, you start with some random vector. Say 1, 0, 1.

So, it has to be linearly independent. So, I need to choose a vector that is linearly independent to the existing one. So, that's just not a constant multiple of that one. So, I can make this a 0, 1, 0. This way for sure this is not a constant multiple of that one. OK. And if you look at just so far the span of those two. Let's call those v1 and call this v2. If you now look at the span of v1 and v2, do you get the entire R3? OK. So, how do you actually answer that question? Well, you would look at the reduced echelon form of the matrix formed by these two to see if you get a pivot every row. Well, because this has only got two columns, at most you're going to get two pivots. So, at most you're going to get a pivot here and there. So, that forces you to have a matrix of this form. This is just maybe some other stuff here. OK? And so pivots in two rows. That leaves that last row of non-pivots, therefore there is no way you can span the entire space because this is not going to be on two map.

So, if it's not on two, you can't get the entire [inaudible]. OK. So, right away we can answer this question. This is no. You can't span. So, it's still not spanning. We have to add one more vector to it. Well, I have to also make sure that the element that I add to the existing set keeps the linearly independence going. OK? So, I want to choose something that is not a constant
multiple of each of those or a linear combination of those. I could start with kind of just using the linear combination. I can just take maybe one of the v1 and then minus 1 v2. So, what does that give me? That would be a 1, negative 1, and 1. So, I know this is a linear combination, right? Because I use the v coefficient of 1 and negative 1 here. But I can add just some extra stuff to a particular component. So, I'm going to add maybe 2 here.

So, I'm going to get a vector 1, negative 1, 3 here. And you can verify that, that this vector that you've just created using this trick would be linearly independent to the existing set. The best way to check is to do the reduced echelon form to see if you get a pivot every row. Because if you do get a pivot every row then for sure the span is R3. If you get a pivot every column it tells you that map is 1 to 1 therefore it has to be a linearly independent set. So, hopefully you have verified it with your calculator that this is indeed the identity matrix. So, not only that, it spans and also is linearly independent. So, this set right here that we've just formed is a basis. Now, I want to ask you the question. So, I just found another basis for R3. So, you've noticed that the basis is certainly not a unique thing.

There are many, many basis for a space. But which one is better in a sense that it's more compact and it's looking simple and so on? So, we've got this basis right here. And I've got a basis before the i, j, k. Which one is better and why? I would say that i, j, k actually is a better basis for a couple of reasons. First of all, it was so easy for me to verify that it is a 1 to 1 and [inaudible] map, which then spans the space and linearly independent. Much, much easier than to determine that for this set right here.

And a very important fact is that it was so easy for me to write linear combinations of a general vector simply by just using the basic [inaudible] vector itself to come up with the correct coefficient. The reason why i, j, k is such a nice set is because they're perpendicular to each other. They're mutually perpendicular to each other. And that's a really, really nice property. Another very nice property of it is because they are all of unit length. So, in conclusion, there are many basis for space, and in general we always like to come up with the basis that is the simplest. And that is called the standard basis for that space. So, in this case, the i, j, and k forms a standard basis for the set R3.