CSULB MATH DAY AT THE BEACH 2019 SOLUTIONS

INDIVIDUAL ROUND, PART 1: MULTIPLE CHOICE

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1. How many real numbers x are such that $\sqrt{2 - x/2} = 2 + x$?

   (A) 0 (B) 1 (C) 2 (D) 3 (E) 4

   Answer: (B) 1

   Solution: $2 - \frac{x}{2} = (2+x)^2 \Leftrightarrow 2 - \frac{x}{2} = 4 + 4x + x^2 \Leftrightarrow x^2 + \frac{9}{2}x + 2 = 0$. There are two solutions to the quadratic equation $x = -4$ and $x = -1/2$, but only $x = -1/2$ solves the original equation with the radical.

   Solved by 92 out of the 221 contestants (41.6%).

2. A box contains four different pairs of socks. Socks are drawn randomly, two at a time, without replacement, until all eight socks are drawn. What is the chance that all four pairs match?

   (A) 1/8 (B) 1/20 (C) 1/105 (D) 1/24 (E) 1/144

   Answer: (C) 1/105

   Solution: Draw two socks randomly out of a box with 8 socks. The chance that the second matches the first equals 1/7. Repeat this procedure with 6 socks left, etc. So, the chance that all four pairs match is (1/7)(1/5)(1/3)(1/1)=1/105.

   Solved by 130 out of the 221 contestants (58.8%).

3. What values of x satisfy the inequality $\log_2(x+1) > \log_2 3 + \log_4 25 + \log_8 343$?

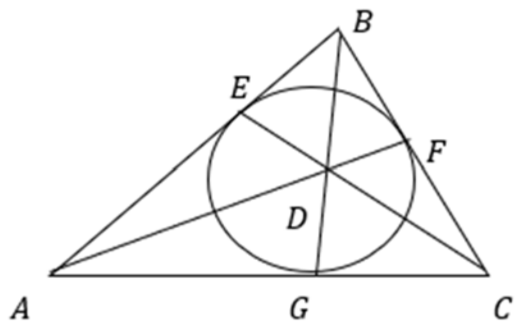
   (A) $x \in (104, \infty)$ (B) $x \in \left(-\infty, \frac{7}{22}\right]$ (C) $\left(\frac{4}{3}, \frac{6}{5}\right)$ (D) $(21, \infty)$ (E) $[0, 1]$

   Answer: (A) $x \in (104, \infty)$

   Solution: Converting all the terms to logarithm with base 2, we get $\log_2(x+1) > \log_2 3 + \log_4 2 \cdot \log_2 25 + \log_8 2 \cdot \log_2 343 = \log_2 3 + \frac{1}{2}\log_2 25 + \frac{1}{3}\log_2 343 =$ $\log_2 3 + \log_2 5 + \log_2 7 = \log_2 105$, thus, $x + 1 > 105$ or $x > 104$.

   Solved by 138 out of the 221 contestants (62.4%).

4. A circle with center D is inscribed in $\triangle ABC$, $\angle EAD = 15°$ and $\angle GCF = 76°$. Find $\angle DBF$.



(A) 32° (B) 40° (C) 37° (D) 74° (E) 18°

Answer: (C) 37°

Solution: Since the point D is the center of the inscribed circle, the ray AD bisects the angle A. Therefore, $\angle A = 2\angle EAD = 30°$, and $\angle DBF = \frac{1}{2}\angle B = \frac{1}{2}(180° - \angle A - \angle C) = \frac{1}{2}(180° - 30° - 76°) = \frac{1}{2}74° = 37°$.

Solved by 86 out of the 221 contestants (38.9%).

5. Positive integers a and b satisfy $\frac{a}{b} = 0.2019\ldots$. What is the smallest possible value of b?

(A) 7 (B) 20 (C) 2018 (D) 104 (E) 1104

Answer: (D) 104

Solution: We have $0.2019 \leq \frac{a}{b} \leq 0.2020$. Therefore, $4.950495 \cdot a \leq b \leq 4.952947 \cdot a$.

For all values of a below 21, the bounds don't contain an integer. For $a = 21, 103.96 \leq b \leq 104.01$. Thus, the smallest possible value of b is 104.

Solved by 79 out of the 221 contestants (35.7%).

6. How many solutions of the equation $\sin^4 x + \cos^4 x = 1$ are there in the interval $0 \leq x \leq 2\pi$?

(A) 0 (B) 1 (C) 2 (D) 4 (E) 5

Answer: (E) 5

Solution: $\sin^4 x + \cos^4 x - 1 = (\sin^2 x + \cos^2 x)^2 - 2\sin^2 x \cos^2 x - 1 = -\frac{1}{2}\sin^2(2x)$, thus the given equation is equivalent to $sin(2x) = 0$, which roots in the given interval are $x = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$, and 2π.

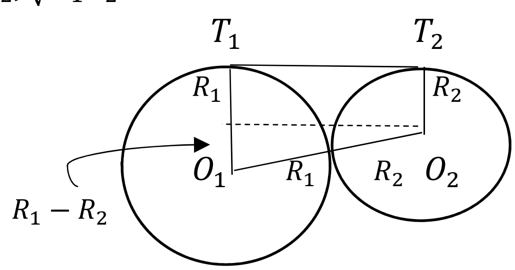
Solved by 85 out of the 221 contestants (38.5%).

7. Two circles of radii R_1 and R_2 are touching externally and have the centers at the points O_1 and O_2. A common tangent line touches the circles at points T_1 and T_2, respectively. Find the area of the polygon $O_1T_1T_2O_2$.

(A) $\frac{1}{2}R_1R_2$ (B) $(R_1 + R_2)^2$ (C) $\frac{1}{2}(R_1 + R_2)\sqrt{R_1R_2}$ (D) $(R_1 + R_2)\sqrt{R_1R_2}$ (E) $R_1 + 2R_2$

Answer: (D) $(R_1 + R_2)\sqrt{R_1R_2}$

Solution: The polygon $O_1T_1T_2O_2$ is a trapezoid with the parallel sides of lengths R_1 and R_2, and the height $|T_1T_2| = \sqrt{(R_1 + R_2)^2 - (R_1 - R_2)^2} = 2\sqrt{R_1R_2}$. Thus, the area is $\frac{R_1+R_2}{2} \cdot 2\sqrt{R_1R_2} = (R_1 + R_2)\sqrt{R_1R_2}$.



Solved by 127 out of the 221 contestants (57.5%).

8. Choose three numbers from the set $\{1, 2, 3, \dots, 20\}$ so that no two numbers are consecutive. How many different choices are there?

(A) 816 (B) 1298 (C) 905 (D) 2019 (E) 1116

Answer: (A) 816

Solution: The middle number x can be chosen between 3 and 18; the smallest number, between 1 and $x - 2$; and the largest number, between $x + 2$ and 20, independently of the smallest number. Thus, for a fixed x, there are $(x - 2)(19 - x)$ choices. Overall, the number of choices is $\sum_{x=3}^{18}(x - 2)(19 - x) = \{i = x - 2\} = \sum_{i=1}^{16} i(17 - i) = 17\sum_{i=1}^{16} i - \sum_{i=1}^{16} i^2 = 17\frac{17 \cdot 16}{2} - \frac{16 \cdot 17 \cdot (2 \cdot 16+1)}{6} = 816$.

Solved by 67 out of the 221 contestants (30.3%).

9. Letters $A, B, C,$ and D denote distinct digits between 0 and 9. The sum of four-digit numbers $BACA$ and $CCCC$ equals $DDBAC$. What is the value of $A + B + C + D$?

(A) 8 (B) 12 (C) 15 (D) 18 (E) 13

Answer: (B) 12

Solution:

+ BACA
CCCC
DDBAC

Looking at the column of ones, we can see immediately that $A = 0$, and thus, from the column of tens, $C = 5$. So, we have the sum $B050 + 5555 = DDB05$. Hence, $B = 6, D = 1$, and $A + B + C + D = 0 + 6 + 5 + 1 = 12$.

Solved by 161 out of the 221 contestants (72.9%).

10. Let $\{x_n\}$ be a sequence such that $x_1 = 2$, and $(n+1)x_{n+1} = x_n + n$ for all $n \geq 1$. What is x_{2019} ?

(A) $\frac{2018!}{2018!+2}$ (B) $\frac{2019!}{2019!+1}$ (C) $\frac{2019!+1}{2019!}$ (D) $\frac{2018!+1}{2018!}$ (E) $\frac{2018!}{2018!+1}$

Answer: (C) $\frac{2019!+1}{2019!}$

Solution: It can be shown by induction that $x_n = 1 + \frac{1}{n!}$. Indeed, it is true for $x_1 = 1 + \frac{1}{1!}$. Suppose that it is true for any $n > 1$, that is, $x_n = 1 + \frac{1}{n!}$. Now, $x_{n+1} = \frac{x_n+n}{n+1} = \frac{1+\frac{1}{n!}+n}{n+1} = 1 + \frac{1}{(n+1)!}$.

Thus, $x_{2019} = 1 + \frac{1}{2019!} = \frac{2019!+1}{2019!}$.

Solved by 126 out of the 221 contestants (57.0%).

INDIVIDUAL ROUND, PART 2: FREE RESPONSE

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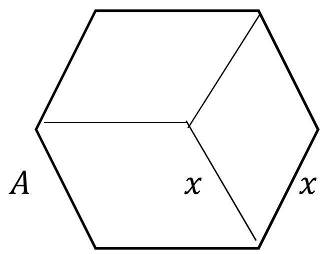
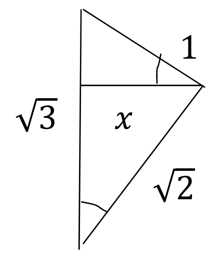
11. Let C be a cube with side length of 1. We put C in the Euclidean 3-dimensional space so that one of its main diagonals is contained in the z-axis. Let A be the projection of C to the xy-plane (that is, A is the set of points (x, y) so that there exists z with $(x, y, z) \in C$). What is the area of A?

Answer: $\sqrt{3}$

Solution: A is a hexahedron (see Picture 1) with the sides x that can be found from the triangle in Picture 2. From similarity of triangles, $\frac{\sqrt{2}}{\sqrt{3}} = \frac{x}{1}$, thus, $x = \sqrt{\frac{2}{3}}$ and the area of A is equal to $6 \cdot \frac{\sqrt{3}}{4} \cdot x^2 = \sqrt{3}$.



Picture 1.



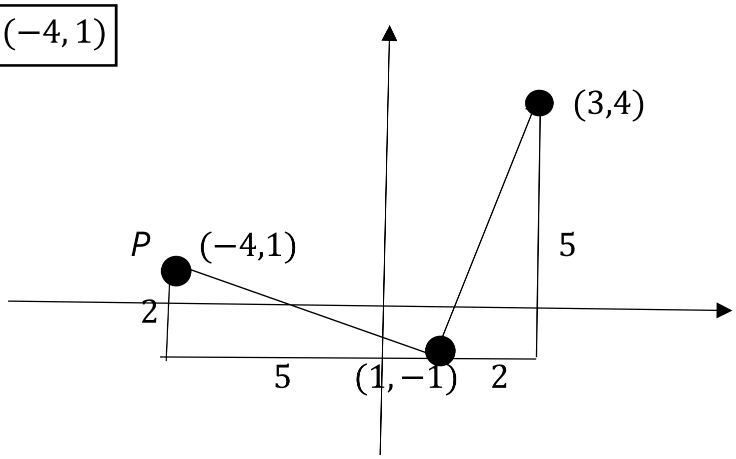
Picture 2.

Solved by 9 out of the 221 contestants (4.1%).

12. Let P be the point obtained by rotating the point (3, 4) around the point (1, -1) counterclockwise by 90°. Find the coordinates of P.

Answer: $\boxed{(-4, 1)}$

Solution:



As seen in the picture, the two right triangles must be congruent, which determines the coordinates of the point P as $(-4,1)$.

Solved by 143 out of the 221 contestants (64.7%).

13. How many integers n with $2 \leq n \leq 750$ can be written as $n = a^b$, so that a and b are integers, and $b > 1$?

Answer: 35

Solution: $b = 2$, $a^b = 2^2, 3^2, 4^2, \ldots, 27^2$ (26 integers); $b = 3, a^b = 2^3, 3^3, 4^3, \ldots, 9^3$ (8 integers); $b = 4, a^b = 2^4, 3^4, 4^4, 5^4$ (4 integers); $b = 5, a^b = 2^5, 3^5$ (2 integers); $b = 6, a^b = 2^6, 3^6$ (2 integers). The total number of integers is $26 + 8 + 4 + 2 + 2 = 42$,

but $16 = 4^2 = 2^4$, $81 = 9^2 = 3^4$, and $512 = 2^9 = 8^3$ are counted twice, and $64 = 8^2 = 4^3 = 2^6$, and $729 = 27^2 = 9^3 = 3^6$ are counted three times, so there are 42-7=35 distinct such integers.

Solved by 42 out of the 221 contestants (19.0%).

14. What is the closest integer to $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{2019}$?

Answer: 8

Solution: $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{2019} \approx \ln 2019 \approx 7.6$, which rounds to 8.

Solved by 26 out of the 221 contestants (11.8%).

INDIVIDUAL ROUND, PART 3: FREE RESPONSE

15. Factorial of what number should be removed from the product $(1!)(2!)(3!) \ldots (20!)$ to make the remaining product a square of a positive integer?

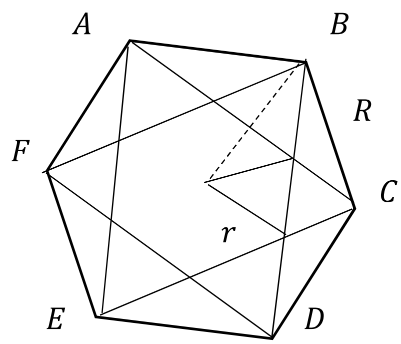
Answer: 10

Solution: The product $(1!)(2!)(3!) \ldots (20!) = 2^{168}3^{78}5^{34}7^{21}11^{10}13^{8}17^{4}19^{2}$.

The factorial $10! = 2^8 3^4 5^2 7^1$. Thus, the ratio $(1!)(2!)(3!) \ldots (20!)/10!$ has all even powers of the prime factors, and by direct verification, 10 is the only solution.

Solved by 51 out of the 221 contestants (23.1%).

16. In a regular hexagon $ABCDEF$ of area 1, the vertices are connected as shown on the picture. Find the area of the smaller hexagon formed in the middle.



Answer: 1/3

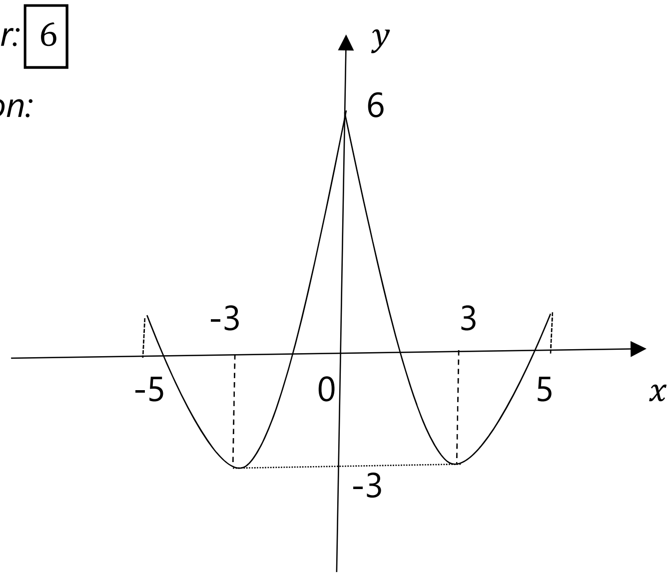
Solution: Let R be the length of a side of the bigger hexagon, and let r be the length of the side of the smaller one. The area of the bigger hexagon is $6 \cdot \frac{1}{2}R \cdot \frac{\sqrt{3}}{2}R = \frac{3\sqrt{3}}{2}R^2 = 1$. The area of the smaller hexagon is $\frac{3\sqrt{3}}{2}r^2$, where $r = \frac{2}{3} \cdot \frac{\sqrt{3}}{2}R = \frac{1}{\sqrt{3}}R$. Thus, the area of the smaller hexagon is $\frac{3\sqrt{3}}{2}\left(\frac{1}{\sqrt{3}}R\right)^2 = \frac{1}{3}$.

Solved by 94 out of the 221 contestants (42.5%).

17. Find the maximum of the function $y = x^2 - 6|x| + 6$ in the interval $-5 \le x \le 5$.

Answer: 6

Solution:



As seen in the graph, the maximum of the function is achieved at $x = 0$, and the maximum value is 6.

Solved by 166 out of the 221 contestants (75.1%).

18. Popcorn is packaged in boxes before a baseball game, and in each box the vendor inserts a card of one of the five recent most valuable players (MVPs), picked randomly. You buy four boxes of popcorn. What is the probability that you get the cards for exactly three different MVPs?

Answer: $\boxed{360/625 = 72/125 = 0.576}$
Solution: There are $5^4 = 625$ ways to put the five cards into the four boxes. We need to have exactly three different cards in those four boxes. There are $\binom{5}{3}$ ways to choose the three cards from among the five cards. Denote the three cards $A, B,$ and C. There are three possibilities for the fourth card: $A, B,$ or $C,$ and $\frac{4!}{2!1!1!}$ ways to place them in the four boxes. Hence, there are a total of $\binom{5}{3} \cdot 3 \cdot \frac{4!}{2!1!1!} = 10 \cdot 3 \cdot 12 = 360$ ways to see exactly three different cards in the four boxes, and the probability is 360/625.

As an additional exercise, we can find the entire distribution of the number of different cards in the four boxes. There could possibly be 1, 2, 3, or 4 cards. The probabilities are: $P(1\ card) = \frac{1}{625} \cdot \binom{5}{1} = \frac{5}{625}$, $P(2\ cards) = \frac{1}{625} \cdot \binom{5}{2} \cdot 2^2 \cdot \frac{4!}{2!2!} = \frac{240}{625}$, $P(3\ cards) = \frac{360}{625}$, and $P(4\ cards) = \frac{1}{625} \cdot \binom{5}{4} \cdot 4! = \frac{120}{625}$.

Solved by 38 out of the 221 contestants (17.2%).

TEAM ROUND

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1. How many ordered pairs (a, b) are such that the equation $x^4 - (a + b + 1)x^3 + (a + b + ab)x^2 - abx = 0$ has exactly two distinct solutions?

   Answer: $\boxed{4}$
   Solution: The polynomial $x^4 - (a + b + 1)x^3 + (a + b + ab)x^2 - abx$ factors as $x(x - 1)(x - a)(x - b)$. Hence, the solutions of the given equation are $0, 1, a,$ and b. Since we want exactly two distinct solutions, (a, b) can be $(0, 0), (1, 0), (0, 1)$, or $(1, 1)$.

   Solved by 10 out of the 38 teams (26.3%).

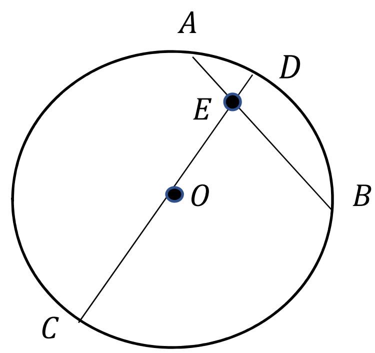
2. If $0 \le x \le \frac{\pi}{2}$, in what interval does the inequality $\cos^3 x < 3\sin^2 x \cdot \cos x$ hold?

Answer: $\boxed{\frac{\pi}{6} < x < \frac{\pi}{2}}$

Solution: $\cos^3 x - 3\sin^2 x \cdot \cos x = \cos x(\cos^2 x - \sin^2 x) - 2\sin^2 x \cdot \cos x = \cos x \cdot \cos(2x) - \sin x \cdot \sin(2x) = \cos(3x) < 0$ when $\frac{\pi}{2} + 2\pi k < 3x < \frac{3\pi}{2} + 2\pi k, k = 0,1, \ldots.$ Since we are restricted to $\left[0, \frac{\pi}{2}\right]$, the inequality holds in the interval $\frac{\pi}{6} < x < \frac{\pi}{2}$.

Solved by 19 out of the 38 teams (50.0%).

3. In a circle with center O, the cord $\overline{AB}$ intersects the diameter $\overline{CD}$ at the point E. The angle $\angle AOB$ equals 60°, and the ratio of the lengths $|\overline{CE}|$ and $|\overline{ED}|$ is 19:1. Assuming $|\overline{AE}| < |\overline{EB}| = 19$, find $|\overline{AE}|$. Leave the answer in the form $a + b\sqrt{c}$ for some real a, b, and c.



Answer: $\boxed{31 - 10\sqrt{6}}$

Solution: The following equalities hold: $|\overline{CE}| \cdot |\overline{ED}| = |\overline{AE}| \cdot |\overline{EB}| = 19|\overline{AE}|$, $|\overline{CE}| / |\overline{ED}| = 19$, and $|\overline{CE}| + |\overline{ED}| = 2R = 2(|\overline{AE}| + |\overline{EB}|) = 2(|\overline{AE}| + 19)$. Thus, $|\overline{AE}|$ is the solution of the equation $|\overline{AE}| - 10 \cdot \sqrt{|\overline{AE}|} + 19 = 0$. Taking into account that $|\overline{AE}| < 19$, we have that $|\overline{AE}| = 31 - 10\sqrt{6}$.

Solved by 13 out of the 38 teams (34.2%).

4. The numbers 2019^k are divided by 7 for $k = 1,2, \ldots, 2019$. What is the sum of all residuals?

Answer: $\boxed{7{,}067}$

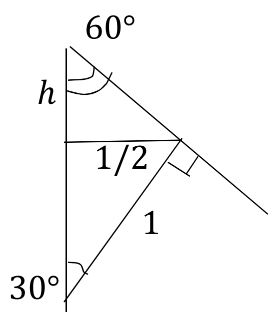
Solution: For $k = 1, 2, 3, 4, 5, 6, 7$, the residuals are $3, 2, 6, 4, 5, 1, 3$. They form a periodic sequence with period 6. Since $2019 = 336 \cdot 6 + 3$, the sum of all residuals is $336 \cdot (3 + 2 + 6 + 4 + 5 + 1) + 3 + 2 + 6 = (336)(21) + 11 = 7{,}067$.

Solved by 21 out of the 38 teams (55.3%).

5. Consider a unit sphere and a point O that lies outside the sphere. Three tangent lines from the point O touch the sphere at the points A, B, and C. The plane ABC intersects the sphere at a circle of radius ½. What is the maximum possible volume of the tetrahedron $OABC$?

Answer: $\boxed{\frac{1}{32}}$

Solution: The volume is maximum if ABC form an equilateral triangle. The triangle is inscribed in a circle of radius $\frac{1}{2}$ and thus has area $\frac{3\sqrt{3}}{4}(\frac{1}{2})^2 = \frac{3\sqrt{3}}{16}$. As seen in the picture, the height of the tetrahedron is $h = \frac{\frac{1}{2}}{tan60°} = \frac{1}{2\sqrt{3}}$. The volume of the tetrahedron is $\frac{1}{3} \cdot h \cdot Area \ \triangle ABC = \frac{1}{3} \cdot \frac{1}{2\sqrt{3}} \cdot \frac{3\sqrt{3}}{16} = \frac{1}{32}$.



Solved by 11 out of the 38 teams (28.9%).

6. Let $\mathbb{R}$ denote the set of real numbers. For what values of $x \in \mathbb{R}$ does the inequality $4\cos^4 x + \sin^2 x \geq 3\cos^2 x$ hold?

Answer: $\boxed{x \in \mathbb{R}}$

Solution: $4\cos^4 x + \sin^2 x \geq 3\cos^2 x \iff 4\cos^4 x + 1 - \cos^2 x \geq 3\cos^2 x \iff 4\cos^4 x - 4\cos^2 x + 1 \geq 0 \iff (1 - 2\cos^2 x)^2 \geq 0$, which is true for any real number x.

Solved by 23 out of the 38 teams (60.5%).

7. All the three-digit numbers are written in a row: 100101102 ... 998999. How many times a two is followed by a zero in this sequence of digits?

Answer: $\boxed{19}$

Solution: A straightforward list is: 120, 200, 201, 202, 203, 204, 205, 206, 207, 208, 209, 220, 320, 420, 520, 620, 720, 820, and 920, which totals 19 times. One can also enumerate all possibilities by noticing that when 2 is in the first position, it must be followed by a zero, which gives 10 cases (the third digit is 0 through 9); if 2 is in the second position, followed by a zero, in the first position there could be digits 1 through 9. So, the total is 10+9=19.

Solved by 31 out of the 38 teams (81.6%).

8. Represent $\frac{2}{7}$ as the sum of two fractions $\frac{1}{a}$ and $\frac{1}{b}$ where a and b are natural numbers with $a < b$.

Answer: $\boxed{\frac{1}{4}+\frac{1}{28}}$
Solution: To satisfy your curiosity, we show that this representation is unique. Suppose a and b are such that $\frac{1}{a}+\frac{1}{b}=\frac{2}{7}$ and $a < b$. Equivalently, $\frac{a+b}{ab}=\frac{2}{7}$, or $a+b=2k$ and $ab=7k$ for some positive integer k. Eliminating k from the two equations, we get the relation $a=\frac{7b}{2b-7}$. Since $a < b$, $\frac{7b}{2b-7} < b$, and so $b > 7$. Trying all integers above 7, we see that $a=4$, and $b=28$ works. Since $a \to \frac{7}{2}=3.5$ when $b \to \infty$, there will be no other whole-number solution with $b > 28$.

Solved by 32 out of the 38 teams (84.2%).

RELAY ROUNDS

Relay 0, Question 1. Compute $4 \times 4 + 4$.

Answer: $\boxed{20}$

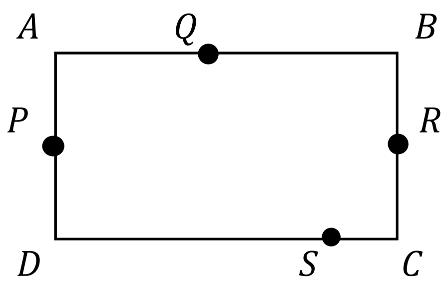
Relay 0, Question 2. Let T=TNYWR. Compute $3T$.

Answer: $\boxed{60}$

Relay 0, Question 3. Let T=TNYWR. Compute $T/6$.

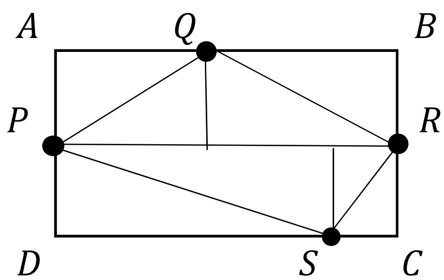
Answer: $\boxed{10}$

Relay 1, Question 1. The points $P, Q, R,$ and S are placed on the sides of rectangle $ABCD$ in such a way that $AP = PD$ and $BR = RC$. The area of the polygon $PQRS$ equals 9. Find the area of $ABCD$.



Answer: 18

Solution:



Area of $ABCD = 2 \cdot |AP| \cdot |PR| = 2 \cdot \text{area of } PQRS = 18.$

Relay 1, Question 2. Let T=TNYWR. Suppose m and n are prime numbers such that $m + n = T - 6$. Find $m^2 + n^2$.

Answer: 74

Solution: $T = 18, m + n = 12,$ so $m = 5, n = 7$, and $m^2 + n^2 = 25 + 49 = 74$.

Relay 1, Question 3. Let T=TNYWR. The area of a rhombus is $T + 6$. If one diagonal is 20 units long, what is the length of the other diagonal?

Answer: 8

Solution: $T = 74$, area of rhombus $= 74 + 6 = 80 = 4 \cdot \frac{20}{2} \cdot \frac{x}{2} \cdot \frac{1}{2}$, so $x = 8$.

Relay 2, Question 1. How many positive integer divisors does 1768 have?

Answer: 16

Solution: $1768 = 2^3 \cdot 13 \cdot 17$, number of divisors is $4 \cdot 2 \cdot 2 = 16$.

Relay 2, Question 2. Let T=TNYWR. How many solutions does the equation $\sin(Tx) + \cos(Tx) = 0$ have in the interval $0 < x < \frac{\pi}{4}$?

Answer: $\boxed{4}$

Solution: $\sin(Tx) + \cos(Tx) = \frac{\sqrt{2}}{2} \cdot \sin\left(\frac{\pi}{4} + Tx\right) = 0$, thus, $\frac{\pi}{4} + Tx = \pi \cdot k, k = 0, 1, 2, \dots,$ and $T = 16$. Since $0 < x < \frac{\pi}{4}$, it must be that $0 < 4k - 1 < 16$ or $k = 1, 2, 3,$ or 4.

Relay 2, Question 3. Let T=TNYWR. What is the sum of the negative integers that satisfy the inequality $2x - 3 > -T - 7$?

Answer: $\boxed{-6}$

Solution: $T = 4$, so x satisfies $2x - 3 > -11$ or $x > -4$ or $x = -3, -2,$ or -1 and the sum of these values is -6.

FACEOFF ROUND

Question 1. What is the largest prime factor of $3 \times 5 \times 7 \times 11 + 2$?

Answer: $\boxed{89}$

Solution: $3 \times 5 \times 7 \times 11 + 2 = 1157 = 13 \cdot 89$.

Question 2. Consider a regular n-sided polygon. If the product of the number of diagonals and the measure (in degrees) of its angle equals 1800, what is the value of n?

Answer: $\boxed{7}$

Solution: The number of diagonals of n-sided polygon is $\frac{n(n-3)}{2}$. The measure of one angle is $\frac{(n-2)\cdot 180}{n}$. The product $\frac{n(n-3)}{2} \cdot \frac{(n-2)\cdot 180}{n} = 1800$, from where $n = 7$.

Question 3. Compute base b if $\log_b 405 = 3 + \log_b 3 + \log_b 5$?

Answer: $\boxed{3}$

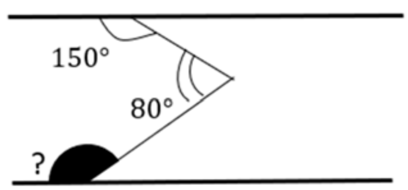
Solution: We have that $15b^3 = 405$, or $b = 3$.

Question 4. Three students are taking a test. Each of them will pass the test with probability 0.7, independently of the others. What is the probability that at least one of them will pass?

Answer: $\boxed{0.973}$

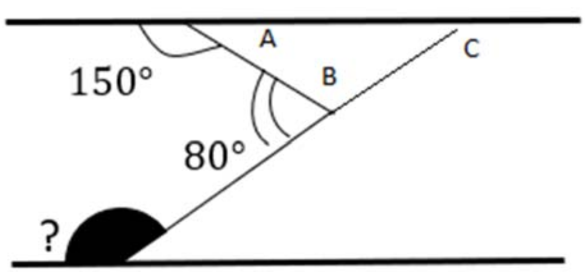
Solution: P(at least one passes)=1-P(none passes)=$1-(0.3)^3=0.973$.

Question 5. The two lines are parallel. Given the two angles, find the third one.



Answer: $\boxed{130°}$

Solution: The unknown angle is equal to $C=180-(180-A-B)=A+B=180-150+180-80=30+100=130°$.



Question 6. Find $m^3+\frac{1}{m^3}$ if $m+\frac{1}{m}=3$.

Answer: $\boxed{18}$

Solution: $(m+1/m)^3=m^3+\frac{1}{m^3}+3(m+\frac{1}{m})$, so $m^3+\frac{1}{m^3}=3^3-3^2=18$.

Question 7. What is the smallest integer greater than 2 such that the division by each of $3, 4,$ and 5 yields a remainder of 2?

Answer: $\boxed{62}$

Solution: Since $3, 4,$ and 5 are relatively prime, their least common multiple is 60, and the smallest integer in question must be 62.

Question 8. Randomly choose a three-digit number (that is, n such that $100 \le n \le 999$). What is the probability that the digits of this number are in a strictly increasing order? Express your answer as a fraction in lowest terms.

Answer: $\boxed{\frac{7}{75}}$

Solution: The three-digit numbers with digits in increasing order are
$123, 124, \dots, 129, 134, \dots, 139, 145, \dots, 149, \dots., 189, 234, \dots, 289, 345, \dots, 389, \dots, 789.$

There are a total of $1 \cdot 7 + 2 \cdot 6 + 3 \cdot 5 + 4 \cdot 4 + 5 \cdot 3 + 6 \cdot 2 + 7 \cdot 1 = 84$ such numbers.

The probability is $\frac{84}{900} = \frac{7}{75}$.

Question 9. Instead of walking in a straight line, a person walked in a perfect semicircle. What percent <u>more</u> of the distance did he walk? Round your answer to two decimals.

Answer: $\boxed{57.08\%}$

Solution: The person walked $\left(\frac{\pi}{2} - 1\right) \cdot 100\% = 57.08\%$ more.

Question 10. Define a function f on the set of positive integers by

$$f(n) = \begin{cases} n - 1 & if\ n\ is\ even, \\ n^2 - 1 & if\ n\ is\ odd. \end{cases}$$

Find and list all n such that $f(f(n)) = 8$.

Answer: $\boxed{4}$

Solution: If n is even, then n solves $8 = f\big(f(n)\big) = f(n-1) = (n-1)^2 - 1$, $n = 4$.

If n is odd, then n solves $8 = f\big(f(n)\big) = f(n^2 - 1) = n^2 - 1 - 1$, or $n^2 = 10$, which has no integer solution.

Question 11. There are 2019 balls. 2018 of them weigh 1 gram each, and one of them weighs 2 grams. You have an electric scale that will tell you the total weight of whatever you put on the scale. How many weightings must you perform, at minimum, to be certain to find the heavy ball?

Answer: $\boxed{11}$

Solution: Note that if we weigh $x$balls, then we know whether the weight is x grams or $x + 1$ grams, and thus can tell if the group of balls contains the heavy one. If you split

2019 balls into two groups of 1009 and 1010, then with one weighting you will know which one contains the heavy ball, and then keep splitting the balls into two groups until we get down to 1 ball. In 11 weightings we will be able to find the heavy ball for sure.

Question 12. Assume $\cos^4\theta + \sin^4\theta = \frac{4}{5}$. Compute $\cos^6\theta + \sin^6\theta$.

Answer: $\boxed{\frac{7}{10}}$

Solution: We have that $\cos^4\theta + \sin^4\theta = (\cos^2\theta + \sin^2\theta)^2 - 2\cdot\cos^2\theta\cdot\sin^2\theta = \frac{4}{5}$. From here, $\cos^2\theta\cdot\sin^2\theta = \frac{1-\frac{4}{5}}{2} = \frac{1}{10}$. Now we write $\cos^6\theta + \sin^6\theta = (\cos^2\theta + \sin^2\theta)^3 - 3\cdot\cos^4\theta\cdot\sin^2\theta - 3\cdot\cos^2\theta\cdot\sin^4\theta = 1 - 3\cdot\cos^2\theta\cdot\sin^2\theta = 1 - \frac{3}{10} = \frac{7}{10}$.