Math Day at the Beach 2018

MULTIPLE CHOICE – *Write your name and school and mark your answers on the answer sheet. You have 30 minutes to work on these problems. No calculator is allowed.*

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1. A bag has some white balls and some red balls. After 7 red balls are removed, the number of white balls becomes 3 times the number of red balls. After 22 white balls are *then* removed, and after that, the number of red balls is 4 times the number of white balls. How many total balls (red plus white) did we start with?

   (A) 15 (B) 24 (C) 29 (D) 39 (E) Cannot be determined from the information given

2. For all x for which the expression is defined, $\dfrac{1}{\sec x + \csc x} =$

   (A) $\dfrac{1}{\cos x + \sin x}$ (B) $\cos x + \sin x$ (C) $\dfrac{\sqrt{2}\sin(2x)}{4\sin(x+\frac{\pi}{4})}$ (D) $\dfrac{\sqrt{2}\sin(2x)}{4\sin(x-\frac{\pi}{4})}$ (E) $\dfrac{\sqrt{2}}{4}$

3. How many distinct real numbers x are there such that

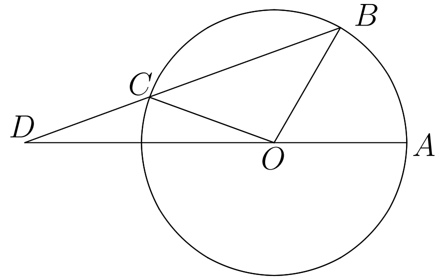
   $$\sqrt{x^2+3} - \sqrt{x^2-3} = \frac{3x}{2}\,?$$

   (A) 0 (B) 1 (C) 2 (D) 3 (E) 4 or more

4. A college math department has 8 different classes that need to be taught and 4 professors available to teach them. If each professor teaches exactly two of the classes, in how many different ways can they be assigned to these classes?

   (A) 70 (B) 256 (C) 330 (D) 1680 (E) 2520

5. See the picture below. Assume $OB = OC = CD$. Assume $\angle AOB = 42°$. Compute (in degrees) $\angle ADB$.



   (A) 14° (B) 18° (C) 21° (D) 24° (E) 48°

6. If x satisfies $x^4 + 2x^3 - 22x^2 + 2x + 1 = 0$ and a, b are the possible values of $x + \frac{1}{x}$, find $a + b$.

   (A) -4 (B) -2 (C) 2 (D) 4 (E) 6

7. In rectangle $ABCD$ with $AB > AD$, a circular arc centered at A with radius AD meets $\overline{AB}$ at E and a circular arc centered at C with radius CD meets $\overline{AB}$ at F. If $AD = 5$, $CD = \frac{17}{3}$, then EF equals:

   (A) 2 (B) 3 (C) $\frac{2}{3}$ (D) $\frac{7}{3}$ (E) $\frac{10}{3}$

8. What is the largest integer k for which 85! is divisible by 42^k?

   (A) 3 (B) 13 (C) 41 (D) 81 (E) 135

9. Let x be the probability that in two flips of a fair coin one is Head and one Tail. Let y be the probability that in four flips of a fair coin two are Heads and two are Tails. Let z be the probability that in five flips of a fair coin three are Heads and two are Tails. Which of the following is the correct listing of the order of these numbers?

   (A) $x < y < z$ (B) $y < z < x$ (C) $z < x < y$ (D) $z < y < x$ (E) $z < x$; $x = y$

10. The number 100! is an integer with k digits (in base 10). Then,

   (A) $100 \leq k < 150$ (B) $150 \leq k < 200$ (C) $200 \leq k < 250$ (D) $250 \leq k < 300$ (E) $300 \leq k < 5050$

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INDIVIDUAL FREE RESPONSE PART 2 – *Write your name and school and mark your answers on your answer sheet. You have 20 minutes to work on these problems. No calculator is allowed.*

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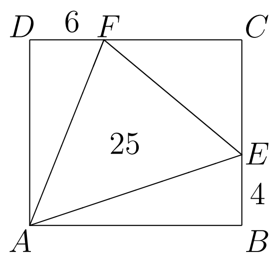
11. Determine all positive real numbers x for which $\log_4 x - \log_x 16 = \frac{7}{6} - \log_x 8$.

12. Suppose $BE = 4\,\text{cm}$, $DF = 6\,\text{cm}$, and the area of $\triangle AEF$ is $25\,\text{cm}^2$. Find the area of rectangle $ABCD$.



13. How many positive integers less then or equal to 2018 are divisible by at least one of $3, 11$, or 61?

14. Form the sequence such that $x_1 = x_2 = 1$, and for $n > 2$, $x_n = x_{n-1}^2 + x_{n-2}$. Of the numbers $x_1, x_2, \ldots, x_{2018}$, how many are divisible by 3?

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Individual Free Response Part 3 – *Write your name and school and mark your answers on your answer sheet. You have 20 minutes to work on these problems. No calculator is allowed.*

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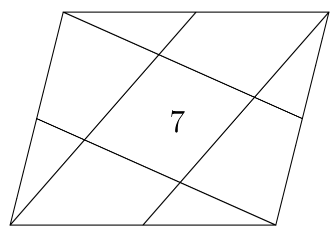
15. There were a total of 555 students in a school. For the next term, the number of girls decreases by 17 and the number of boys increases by $\frac{1}{12}$ of the original number of boys. The total number of students decreases by 11. How many girls are in the school in the new term?

16. Lines from the vertices of a parallelogram to the midpoints of the sides are drawn, as shown, forming a parallelogram of area 7 at the center. What is the area of the original parallelogram?



17. Count the number of integers n so that $1 \leq n \leq 1000000$, and the sum of all digits in n is 11.

18. Three congruent right circular solid cones, each of volume equal to 9, are placed in space so that their bases lie on a given horizontal plane with each one's base touching the other two, and these cones all point upward. A fourth solid cone, congruent to the first three, is turned point down, placed into the space between the first three, and allowed to fall as low as possible. The volume of that portion of the fourth cone that lies below the plane is a number that can be expressed in the form $a\sqrt{b} - c$, where a and c are positive integers and b is a positive square-free integer. Compute $a + b + c$.

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TEAM ROUND – *Write your school name and mark your answers on the answer sheet. No calculator is allowed. You have 30 minutes to work on these problems.*

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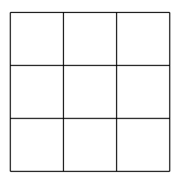
1. Four people are in a room. For each possible grouping of three of them, compute the sum of the average age of the three plus the age of the fourth. The numbers we get this way are $29, 23, 21$, and 17. Find the difference between the age of the oldest of them and the age of the youngest of them.

2. Find all positive values of b for which the series $\sum_{n=1}^{\infty} b^{\ln n}$ converges.

3. If $\sin x + \cos x = \frac{1}{3}$, what is the value of $\sin^3 x + \cos^3 x$?

4. Consider a 3×3 array of squares. Color each square one of three possible colors (red, yellow, or blue) so that no two squares that share an edge have the same color. How many ways are there to do this coloring?



5. Find the minimum value of $f(x, y)$ over all possible real numbers x and y :

$$f(x,y) = \sqrt{4+y^2} + \sqrt{(x-2)^2+(2-y)^2} + \sqrt{(4-x)^2+1}$$

6. In triangle ABC, assume $AB = AC$ and $AC < BC$. Extend $\overline{CA}$ to E so that $BC = CE$. Then extend $\overline{AB}$ to point D such that $AD = DE$. It turns out that we also have $AD = BC$. Compute (in degrees) $\angle ACB$.

7. Six congruent squares are externally tangent to circle C_1, arranged in a nonoverlapping fashion so that each square shares one vertex each with two other squares. Circle C_2 (which shares a center with C_1) is circumscribed about all of the squares. Compute the fraction (area of C_2)/(area of C_1).

8. Define a sequence by $a_0 = 0$, $a_1 = 1$, and for $n \geq 2$, $6a_n = 11a_{n-1} - 5a_{n-2} + 11$. What integer is closest to a_{1000}?