Math Day at the Beach 2017

Multiple Choice – *Write your name and school and mark your answers on the answer sheet. You have 30 minutes to work on these problems. No calculator is allowed.*

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1. How many integers n satisfy $n^4 + 6n < 6n^3 + n^2$?

   (A) 0 (B) 3 (C) 4 (D) 5 (E) 6

2. How many ordered pairs (x, y) of real numbers are solutions to the following system of equations?

$$x^3 - 3xy^2 = 8$$
$$3x^2y - y^3 = 0$$

   (A) 0 (B) 1 (C) 2 (D) 3 (E) 4 or more

3. The minimum value over all real numbers x of the function $f(x) = \max\{\sin x, \cos x\}$ is:

   (A) -1 (B) $-\frac{\sqrt{2}}{2}$ (C) 0 (D) $\frac{\sqrt{2}}{2}$ (E) 1

4. Trains A and B are traveling along parallel tracks in the same direction. Train A is 280 meters long and is traveling at 5 m/sec, while train B is 200 meters long and is traveling at 3 m/sec. Train A is initially behind train B but eventually passes it. Find the length of time during which any part of the two trains overlaps.

   (A) 10 sec (B) 60 sec (C) 90 sec (D) 140 sec (E) 240 sec

5. For positive integers m, n and nonzero real numbers a, b, the graphs of $y = ax^m$ and $y = bx^n$ meet at exactly 3 distinct points. Compute $(-1)^{m+n} + \frac{ab}{|ab|}$.

   (A) -2 (B) 0 (C) 1 (D) 2 (E) Such graphs are impossible

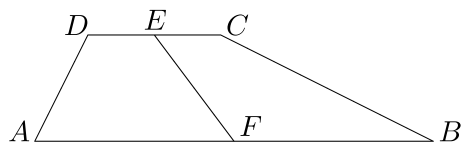
6. Suppose x and y are chosen from the set $\{0, 1, 2\}$. How many ordered pairs (x, y) are there such that there exist integers a and b, not both divisible by 3, with $ax + by$ and $ax^2 + by^2$ both divisible by 3?

   (A) 5 (B) 6 (C) 7 (D) 8 (E) 9

7. What is the smallest integer n so that if you choose *any* n *mutually relatively prime* integers from $\{2, 3, \ldots, 50\}$, at least one of them must be prime?

   (A) 3 (B) 4 (C) 5 (D) 6 (E) 7 or more

8. $ABCD$ is a trapezoid such that $\angle A + \angle B = 90°$, $AB = 30$, and $CD = 10$. Let E be the midpoint of $\overline{CD}$ and F be the midpoint of AB. Compute EF.



   (A) 10 (B) 15 (C) 20 (D) 25 (E) Cannot be determined from the information given.

9. Suppose $p(x)$ is a sixth degree polynomial and an even function. If $p(0) = 0$, $p(1) = 2$, $p(2) = 5$, and $p(3) = 10$, find the units digit (in base 10) of $p(4)$.

   (A) 0 (B) 2 (C) 4 (D) 6 (E) 8

10. Four of the edges of a tetrahedron have length 1. Find the maximum possible volume of this tetrahedron.

   (A) $\frac{1}{6}$ (B) $\frac{\sqrt{3}}{6}$ (C) $\frac{2\sqrt{3}}{27}$ (D) $\frac{\sqrt{3}}{12}$ (E) $\frac{\sqrt{3}}{16}$

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Individual Free Response Part 2 – *Write your name and school and mark your answers on your answer sheet. You have 20 minutes to work on these problems. No calculator is allowed.*

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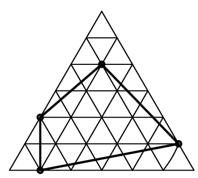
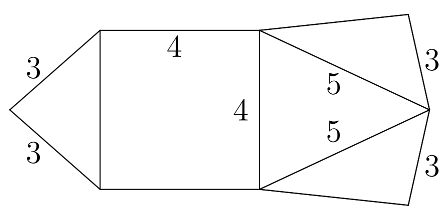
11. Each of the small equilateral triangles below has area 1. Find the area of quadrilateral enclosed by the thick lines.



12. If we cut out the following plane figure and crease it along the lines shown, we can fold it to create an asymmetrical pyramid with a square base. Find the volume of that pyramid.



13. Find the constant term in the expansion of $\left(x^2 + y - \dfrac{1}{xy}\right)^{10}$.

14. Two coins are in a sack. One is a fair coin whose probability of coming up heads is $\dfrac{1}{2}$; the other coin is an unfair coin whose probability of coming up heads is $\dfrac{3}{4}$. One coin is selected from the sack at random and tossed. Given that it came up heads, what is the probability that the same coin will come up heads when tossed a second time?

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Individual Free Response Part 3 – *Write your name and school and mark your answers on your answer sheet. You have 20 minutes to work on these problems. No calculator is allowed.*

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15. ABC is an isosceles right triangle with right angle at C. Circle ω is centered at point D, where D lies on ray $\overrightarrow{CA}$ and circle ω passes through both point C and the midpoint of hypotenuse $\overline{AB}$. A point is selected at random in the interior of $\triangle ABC$. What is the probability that the point lies inside circle ω?

16. Which digit should $x \in \{0, 1, \ldots, 9\}$ be so that the 4035 digit number $\underbrace{55\ldots5}_{\text{2017 times}}x\underbrace{66\ldots6}_{\text{2017 times}}$ is a multiple of 7?

17. Solve the equation for x :
$$\log_{8x} 8 + 2\log_{64x} 8 = 0.$$

18. Three consecutive integers, written in base 5, are ABC_5, ABD_5, AAE_5. Here A, B, C, D, E are the five possible digits $0, 1, 2, 3, 4$ in some permutation. What is ABD_5 in base 10?

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Team Round – *Write your school name and mark your answers on the answer sheet. No calculator is allowed. You have 30 minutes to work on these problems.*

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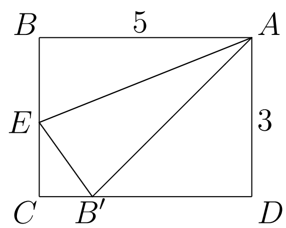
1. Two coins are in a sack. One is a fair coin whose probability of coming up heads is $\frac{1}{2}$; the other coin is an unfair coin whose probability of coming up heads is p, where $p \in (0, 1)$ is a number to be determined later. One coin is selected from the sack at random and tossed. Given that it came up heads, what is the smallest (over all possible p) probability that the same coin will come up heads when tossed a second time?

2. Find the smallest integer n so that $n > 1$ and the product of all of the positive integer divisors of n is n^7.

3. Rectangle $ABCD$ has $AB = 5$ and $AD = 3$. Fold the rectangle creased at AE so that B lands at B' on CD. Find BE.



4. Find two positive integers a and b such that $a^2 + b^2 = 85113$ and the least common multiple of a and b is 1764. Express your answer as an ordered pair (a, b).

5. We have 27 cubes, each $1 \times 1 \times 1$, of which some are transparent and some are opaque. We stack them together to make a $3 \times 3 \times 3$ cube. When projected onto the planes parallel to its faces, this appears as an opaque 3×3 square in each of the three directions. What is the smallest number of opaque cubes needed to do this?

6. Compute this sum:

$$\frac{2^{\frac{1}{1000}}}{2^{\frac{1}{1000}} + 2^{\frac{1}{2}}} + \frac{2^{\frac{2}{1000}}}{2^{\frac{2}{1000}} + 2^{\frac{1}{2}}} + \frac{2^{\frac{3}{1000}}}{2^{\frac{3}{1000}} + 2^{\frac{1}{2}}} + \cdots + \frac{2^{\frac{999}{1000}}}{2^{\frac{999}{1000}} + 2^{\frac{1}{2}}}$$

7. What is the value of $\cos^2 10° + \cos^2 50° - \sin 40° \sin 80°$?

8. Let S be the set of all ordered pairs (x, y) of *nonnegative* integers such that

$$x^3 + y^3 + 21xy = 343.$$

If $S = \{(x_i, y_i) : 1 \le i \le n\}$ is a set with n members, find $\displaystyle\sum_{i=1}^{n} x_i + \sum_{i=1}^{n} y_i$.