# Math Day at the Beach 2017 Solutions

Mike Bao, Brendan Brzycki, Benjamin Chen, Samuel Cui,
Michael Diao, Ayush Kamat, Emma Qin, Jack Sun, Jason Ye,
Xuyang Yu, Beckman Math Club

## 1 Individual Round 1

**Problem 1.1** (Problem 1)
How many integers $n$ satisfy $n^4 + 6n < 6n^3 + n^2$?

**(A)** 0 **(B)** 3 **(C)** 4 **(D)** 5 **(E)** 6

*Solution.* Rearraging things slightly and factoring:

$$n^4 - 6n^3 - n^2 + 6n < 0$$
$$n(n-6)(n^2-1) < 0$$

For $n = -1$, 0, or 1, the left side equals 0, so those values do not work.
For all other $n$, $n^2 - 1 > 0$, so we can reduce the inequality to

$$n(n-6) < 0$$

This means $n$ and $n - 6$ must be opposite signs, so the only values of $n$ that work are 2, 3, 4, and 5, giving us a final answer of $\boxed{\textbf{(C)}\ 4}$.

*Solved by 150 out of 205 contestants* □

**Problem 1.2** (Problem 2)
How many ordered pairs $(x, y)$ of real numbers are solutions to the following system of equations?

$$x^3 - 3xy^2 = 8$$
$$3x^2y - y^3 = 0$$

**(A)** 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4 or more

*Solution.* Factor the second equation as:

$$y(3x^2 - y^2) = 0$$

This gives us two cases: $y = 0$ or $3x^2 = y^2$. First, let $y = 0$. Plugging into the first equation, this forces $x = 2$. Thus, $(2, 0)$ is a solution.

Now, let $3x^2 = y^2$. Substituting into the first equation, we get:

$$\begin{aligned} x^3 - 3x(3x^2) &= 8 \\ -8x^3 &= 8 \\ x &= -1 \end{aligned}$$

Plugging $x = -1$ into $3x^2 = y^2$, we see that $y = \pm\sqrt{3}$, giving us two additional solutions. Thus, we have a total of $\boxed{\textbf{(D) } 3}$ solutions. □

*Solution 2.* Multiplying both sides of the second equation by $i$ and adding it to the first equation, we get:

$$\begin{aligned} x^3 + 3x^2y \cdot i - 3xy^2 - y^3 \cdot i &= 8 \\ (x + y \cdot i)^3 &= 8 \end{aligned}$$

Let $z = x + y \cdot i$. In the complex plane, there are 3 roots for $z^3 = 8$, which are $2$, $-1 + \sqrt{3}i$, and $-1 - \sqrt{3}i$. Since $x$ and $y$ are both real, each root above corresponds with exactly one solution for $(x, y)$.

The three solutions for $(x, y)$ are $(2, 0)$, $(-1, \sqrt{3})$, and $(-1, -\sqrt{3})$, giving a final answer of $\boxed{\textbf{(D) } 3}$.

*Solved by 46 out of 205 contestants* □

**Problem 1.3** (Problem 3)

The minimum value over all real numbers $x$ of the function $f(x) = \max\{\sin x, \cos x\}$ is:

**(A)** $-1$ **(B)** $-\frac{\sqrt{2}}{2}$ **(C)** $0$ **(D)** $\frac{\sqrt{2}}{2}$ **(E)** $1$

*Solution.* We will only consider $0 \le x < 2\pi$ since both $\sin(x)$ and $\cos(x)$ are periodic with period $2\pi$. Also, note that the minimum value must take place for $x \in (\pi, \frac{3\pi}{2})$ since that is the only region on which both $\sin(x)$ and $\cos(x)$ are negative.

Since $\sin(x)$ is monotonically decreasing on this interval and $\cos(x)$ is monotically increasing, the minimum of $f(x)$ will occur at its cusp, or where $\sin(x) = \cos(x)$. This happens when $x = \frac{5\pi}{4}$.

Thus, our answer is $f\left(\frac{5\pi}{4}\right) = \boxed{\textbf{(B) } -\frac{\sqrt{2}}{2}}$.

*Solved by 106 out of 205 contestants* □

**Problem 1.4** (Problem 4)

Trains A and B are traveling along parallel tracks in the same direction. Train A is 280 meters long and is traveling at 5 m/sec, while train B is 200 meters long and is traveling at 3 m/sec. Train A is initially behind train B but eventually passes it. Find the length of time during which any part of the two trains overlaps.

**(A)** 10 sec **(B)** 60 sec **(C)** 90 sec **(D)** 140 sec **(E)** 240 sec

*Solution.* When Train A is right behind Train B, the head of Train A is 200 meters behind the head of Train B; When Train A has just passed Train B, the head of Train A is 280 meters in front of the head of Train B. Train A has therefore moved 200 + 280 = 480 more meters than Train B, at a rate of 5m/s - 3m/s = 2m/s. Therefore it takes a total of $\frac{480\text{m}}{2\text{m/s}} = \boxed{\textbf{(E)}\ 240 \text{ sec}}$ for Train A to completely pass train B, during which parts of the two trains overlap.

*Solved by 133 out of 205 contestants* □

**Problem 1.5** (Problem 5)

For positive integers $m, n$ and nonzero real numbers $a, b$, the graphs of $y = ax^m$ and $y = bx^n$ meet at exactly 3 distinct points. Compute $(-1)^{m+n} + \frac{ab}{|ab|}$.

**(A)** $-2$ **(B)** $-1$ **(C)** 0 **(D)** 2 **(E)** Such graphs are impossible.

*Solution.* For the graphs to meet at exactly three distinct points, we must have $ax^m = bx^n$ at exactly three distinct values of x. WLOG, let $n < m$, giving

$$ax^m - bx^n = 0$$
$$ax^n(x^{m-n} - \frac{b}{a}) = 0$$

Obviously, $x = 0$ is a solution here. So we must have two distinct solutions from $x^{m-n} - \frac{b}{a} = 0$. Note that $x^{m-n}$ is strictly increasing/decreasing unless $m - n$ is a positive even integer, so $m + n$ must also be a positive even integer. Also, let $m - n = 2k$, $x^{2k}$ for $x > 0$ must be positive. Therefore $\frac{b}{a} > 0$ and $ab > 0$.

So we have $(-1)^{m+n} + \frac{ab}{|ab|} = 1 + 1 = \boxed{\textbf{(D)}\ 2}$.

*Solved by 64 out of 205 contestants* □

**Problem 1.6** (Problem 6)

Suppose $x$ and $y$ are chosen from the set $\{0, 1, 2\}$. How many ordered pairs $(x, y)$ are there such that there exist integers $a$ and $b$, not both divisible by 3, with $ax + by$ and $ax^2 + by^2$ both divisible by 3?

**(A)** 5 **(B)** 6 **(C)** 7 **(D)** 8 **(E)** 9

*Solution.* We proceed by casework on the value of $x$. For $x = 0$, we see that any value of $y$ works as we can let $3 \mid b$ and let $3 \nmid x$. This adds three cases. For $x = 1$, we see that $y = 0$ works from the same logic as above, and that $y = 1$ works as we can let $3 \mid a + b$ without breaking anything. However, $y = 2$ does not work, as then we would have to have

$$\begin{aligned} a + 2b \equiv a + 4b \equiv 0 \pmod 3 \\ \implies a \equiv b \equiv 0 \pmod 3, \end{aligned}$$

which wouldn't work. Hence this adds two cases. Finally, for $x = 2$, we see that $y = 0$ works and $y = 1$ doesn't. Now, we see that $y = 2$ works as we can simply let $3 \mid a + b$ again. Hence this adds two cases and our final answer is $3 + 2 + 2 = \boxed{\textbf{(C) } 7}$.

*Solved by 47 out of 205 contestants* □

**Problem 1.7** (Problem 7)

What is the smallest integer $n$ so that if you choose *any $n$ mutually relatively prime* integers from $\{2, 3, \ldots, 50\}$, at least one of them must be prime?

**(A)** 3 **(B)** 4 **(C)** 5 **(D)** 6 **(E)** 7 or more

*Solution.* $n$ must be at least 5, as you can choose the 4 integers $\{4, 9, 25, 49\}$, which are all composite and mutually relatively prime.

Now assume that there exist 5 composite integers bewteen 2 and 50 that are mutually relatively prime. Since they are mutually relatively prime, all of their prime factors must be distinct.

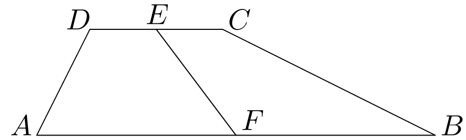
Consider the smallest prime factor for each of the 5 integers. All of these smallest prime factors must be distinct, so by the pigeonhole principle, one of them must have a smallest prime factor of at least 11.

However, the smallest composite integer with a smallest prime factor of 11 is 121, which is much larger than our bounds. Therefore, you cannot have 5 mutually relatively prime composite integers between 2 and 50, so the answer is $n = \boxed{\textbf{(C) } 5}$.

*Solved by 61 out of 205 contestants* □

**Problem 1.8** (Problem 8)

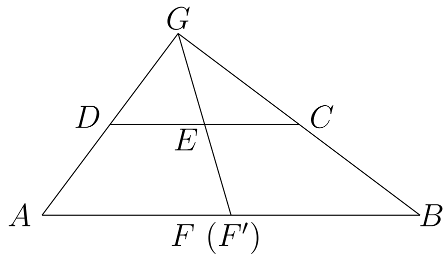
$ABCD$ is a trapezoid such that $\angle A + \angle B = 90°, AB = 30$, and $CD = 10$. Let $E$ be the midpoint of $\overline{CD}$ and $F$ be the midpoint of $AB$. Compute $EF$.




**(A)** 10 **(B)** 15 **(C)** 20 **(D)** 25 **(E)** Cannot be determined.

*Good Solution.*

Diagram:




First, extend $\overline{AD}$ to meet $\overline{BC}$ extended at $G$. Connect and extend $\overline{GE}$ to intersect $\overline{AB}$ at $F'$. Since $\angle A + \angle B = 90°$, $\angle AGB = 180° - 90° = 90°$. Since $E$ is the midpoint, a circle with diameter $\overline{DC}$ centered at $E$ shows that $GE = CE = DE = \frac{10}{2} = 5$. Since $ABCD$ is a trapezoid, $\overline{DC} \parallel \overline{AB}$, then $\frac{DE}{AF'} = \frac{GE}{GF'} = \frac{CE}{BF'}$. Thus $AF' = F'B$, $F$ and $F'$ are the same point. As $F$ (or $F'$) is the midpoint of $AB$, $AF = BF = GF = \frac{30}{2} = 15$. Therefore, $EF = GF - GE = 15 - 5 = \boxed{\textbf{(A) } 10}$.

*Jason Ye's "Solution".* Assume that $EF$ can be determined. Then let $ABCD$ be an isosceles trapezoid.

Let $G$ be the projection from $D$ onto $AB$. Then since $\angle A = \angle B$, both must be 45°, so we know $DG = AG = 10$. Thus we conclude

$$EF = DG = \boxed{\textbf{(A) } 10}.$$

*Solved by 83 out of 205 contestants* □

**Problem 1.9** (Problem 9)

Suppose $p(x)$ is a sixth degree polynomial and an even function. If $p(0) = 0$, $p(1) = 2$, $p(2) = 5$, and $p(3) = 10$, find the units digit (in base 10) of $p(4)$.

**(A)** 0 **(B)** 2 **(C)** 4 **(D)** 6 **(E)** 8

*Solution.* Since $p(x)$ is an even function, it must be of the form $ax^6 + bx^4 + cx^2 + d$. Moreover, since $p(0) = 0$, we know that $d = 0$. We want

$$\begin{aligned} p(4) &\equiv 6(a+b+c) + d \pmod{10} \\ &\equiv 6p(1) \pmod{10} \\ &\equiv 2 \pmod{10}\ . \end{aligned}$$

Hence our final answer is $\boxed{\textbf{(B) } 2}$.

*Solved by 25 out of 205 contestants* □

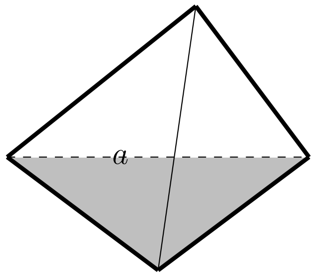
**Problem 1.10** (Problem 10)

Four of the edges of a tetrahedron have length 1. Find the maximum possible volume of this tetrahedron.

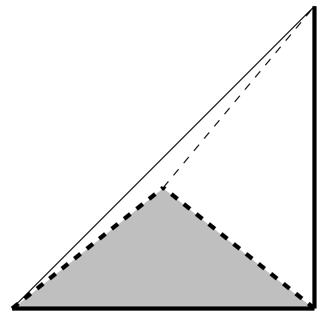
**(A)** $\frac{1}{6}$ **(B)** $\frac{\sqrt{3}}{6}$ **(C)** $\frac{2\sqrt{3}}{27}$ **(D)** $\frac{\sqrt{3}}{12}$ **(E)** $\frac{\sqrt{3}}{16}$

*Solution.* There are only two possible cases.

**Case 1**




**Case 2**



Case 1: Each face of the tetrahedron has at most 2 edges of length 1.

WLOG suppose AB = BC = AD = DC = 1. Let $\triangle ABC$ be the base. $\triangle ADC \cong \triangle ABC$. Given the length of AC, the volume of the tetrahedron is maximized when $\triangle ABC$ is orthogonal to $\triangle ADC$. Let $AC = 2x$. Then $[ABC] = x \cdot \sqrt{1-x^2}$, and the height of the tetrahedron is $\sqrt{1-x^2}$. Therefore the volume is $\frac{1}{3} \cdot x \cdot (1-x^2)$. So... with a bit a calculus you find that the maximum value occurs when $x^2 = \frac{1}{3}$, and the volume is $\frac{2\sqrt{3}}{27}$.

Case 2: One face of the tetrahedron has all three edges of length 1.

Let that face be the base of the tetrahedron, which has area $\frac{\sqrt{3}}{4}$. The height of the tetrahedron can be at most 1 (because some other edge must have length of 1) so the maximun volume in this case is $\frac{1}{3} \cdot \frac{\sqrt{3}}{4} \cdot 1 = \frac{\sqrt{3}}{12}$.

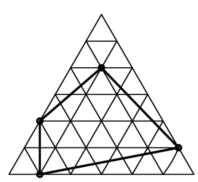
Comparing the two results, we see that $\frac{\sqrt{3}}{12} > \frac{2\sqrt{3}}{27}$, so the answer is $\boxed{\textbf{(D)}\ \frac{\sqrt{3}}{12}}$.

*Solved by 36 out of 205 contestants* □

# 2 Individual Round 2

**Problem 2.1** (Problem 11)

Each of the small equilateral triangles below has area 1. Find the area of the quadrilateral enclosed by the thick lines.

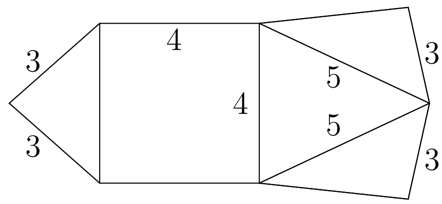


*Solution.* Counting, we see that the total area of the triangular figure is 36. Now we find the area by counting the complement (everything outside the quadrilateral) and subtracting that from 36. We see immediately that there are six triangles that are completely outside the figure, so we add six to the area of our complement. Now we see that we can calculate the area of the region cut off by the thick lines by first computing the area of the unique parallelogram that contains the thick line as a diagonal and dividing by two. Doing this results in some routine calculations that increment our complement by 10. Hence our final complement is 16 and our final answer is $36-16 = \boxed{20}$.

*Solved by 114 out of 205 contestants* □

**Problem 2.2** (Problem 12)

If we cut out the following plane figure and crease it along the lines shown, we can fold it to create an asymmetrical pyramid with a square base. Find the volume of that pyramid.




*Solution.* Since we can fold this figure into a pyramid, we see that the length of the side not given must be 4, as it coincides with the square. Now because of this, two of our triangular faces are $3-4-5$ right triangles, meaning that the vertex of our pyramid is directly above the base of the triangle with side length 3. Calculating the height $h$ of this triangle via the pythagorean theorem gets us that

$$3^2 = 2^2 + h^2 \implies h = \sqrt{5}.$$

Hence using the formula for the volume of a pyramid gives us that our answer is $\frac{1}{3}(4)(4)(\sqrt{5}) = \boxed{\dfrac{16\sqrt{5}}{3}}$.

*Solved by 64 out of 205 contestants* □

**Problem 2.3** (Problem 13)

Find the constant term in the expansion of $\left(x^2 + y - \frac{1}{xy}\right)^{10}$.

*Solution.* Each term in the expansion is in the form of $(x^2)^m \cdot y^n \cdot (\frac{1}{xy})^{10-m-n}$, or $x^{2m-(10-m-n)} \cdot y^{n-(10-m-n)}$. In order for this to be a constant term, we have $2m-(10-m-n) = 0$, $n-(10-m-n) = 0$. We solve for that and get $m = 2, n = 4$. Plugging that in, the constant term is $x^4 \cdot y^4 \cdot x^{-4} \cdot y^{-4} = 1$. By the multinomial theorem, this term appears $\binom{10}{2,4,4}$ times. Therefore the answer is $\frac{10!}{2!\cdot 4!\cdot 4!} = \boxed{3150}$.

*Solved by 44 out of 205 contestants* □

**Problem 2.4** (Problem 14)

Two coins are in a sack. One is a fair coin whose probability of coming up heads is $\frac{1}{2}$; the other coin is an unfair coin whose probability of coming up heads is $\frac{3}{4}$. One coin is selected from the sack at random and tossed. Given that it came up heads, what is the probability that the same coin will come up heads when tossed a second time?

*Solution.* Since we are given that the coin came up heads first, the probability the coin is fair is $\frac{\frac{1}{2}}{\frac{3}{4}+\frac{1}{2}} = \frac{2}{5}$, and the probability that the coin is unfair is hence $1 - \frac{2}{5} = \frac{3}{5}$. Then the probability that the next flip is a head is just the probability of selecting a certain coin multiplied by the probability that that coin will come up heads, summed over both coins, giving an answer of $\frac{3}{5} \cdot \frac{3}{4} + \frac{2}{5} \cdot \frac{1}{2} = \boxed{\frac{13}{20}}$.

*Solved by 63 out of 205 contestants* □

# 3 Individual Round 3

**Problem 3.1** (Problem 15)

$ABC$ is an isosceles right triangle with right angle at $C$. Circle $\omega$ is centered at point $D$, where $D$ lies on ray $\overrightarrow{CA}$ and circle $\omega$ passes through both point $C$ and the midpoint of hypotenuse $\overline{AB}$. A point is selected at random in the interior of $\triangle ABC$. What is the probability that the point lies inside circle $\omega$?

*Solution.* Let the legs of triangle $ABC$ have length $s$. Since circle $\omega$ contains both point $C$ and the midpoint of $\overline{AB}$, its center must be equidistant from these points. However, the center must also be on ray $\overrightarrow{CA}$. The only point that satisfies these two conditions is the midpoint of $\overline{AC}$. Thus, it is easy to see that the area contained in both the circle $\omega$ and the triangle $ABC$ is $\left(\frac{1}{8} + \frac{\pi}{16}\right) s^2$. Since the area of $ABC$ is $\left(\frac{1}{2}\right) s^2$, the desired probability is $\boxed{\frac{2+\pi}{8}}$.

*Solved by 77 out of 205 contestants* □

**Problem 3.2** (Problem 16)

Which digit should $x \in \{0, 1, \ldots, 9\}$ be so that the 4035 digit number $\underbrace{55\ldots5}_{2017 \text{ times}}x\underbrace{66\ldots6}_{2017 \text{ times}}$ is a multiple of 7?

*Solution.* Notice that we can write this obscenely large number as

$$\underbrace{55\ldots5}_{2017 \text{ times}}x\underbrace{66\ldots6}_{2017 \text{ times}} = \underbrace{11\ldots1}_{2017 \text{ times}} \cdot 5 \cdot 10^{2018} + x \cdot 10^{2017} + \underbrace{11\ldots1}_{2017 \text{ times}} \cdot 6.$$

By Fermat's Little Theorem, we have that $10^{2017} \equiv 10 \equiv 3 \pmod 7$. Moreover, by the formula for a geometric series, we can write

$$\underbrace{11\ldots1}_{2017 \text{ times}} = \frac{10^{2017}-1}{9} \equiv \frac{2}{2} \equiv 1 \pmod 7,$$

hence we can rewrite the above as

$$\underbrace{11\ldots1}_{2017 \text{ times}} \cdot 5 \cdot 10^{2018} + x \cdot 10^{2017} + \underbrace{11\ldots1}_{2017 \text{ times}} \cdot 6 \equiv 500 + 10x + 6 \equiv 0 \pmod 7,$$

which simplifies to $x \equiv 4 \pmod 7$. The only base 10 digit which satisfies this is 4, hence $x = \boxed{4}$.

*Solved by 61 out of 205 contestants* □

**Problem 3.3** (Problem 17)
Solve the equation for $x$ :

$$\log_{8x} 8 + 2\log_{64x} 8 = 0.$$

*Solution.* Note that the change of base formula gives that $\log_b a = \frac{1}{\log_a b}$. We use this, along with the other properties of logarithms to get that

$$\begin{aligned} 0 &= \log_{8x} 8 + 2\log_{64x} 8 \\ &= \frac{1}{\log_8 8x} + \frac{2}{\log_8 64x} \\ &= \frac{1}{\log_8 x + 1} + \frac{2}{\log_8 x + 2}. \end{aligned}$$

Making the substitution $y = \log_8 x$ allows us to simplify further:

$$\begin{aligned} 0 &= \frac{1}{y+1} + \frac{2}{y+2} \\ \frac{1}{y+1} &= \frac{-2}{y+2} \\ y+2 &= -2y-2 \\ y &= -\frac{4}{3}. \end{aligned}$$

Hence $\log_8 x = -\frac{4}{3}$ and $x = 8^{-\frac{4}{3}} = \boxed{\frac{1}{16}}$.

*Solved by 69 out of 205 contestants* □

**Problem 3.4** (Problem 18)
Three consecutive integers, written in base 5, are $ABC_5, ABD_5, AAE_5$. Here $A, B, C, D, E$ are the five possible digits $0, 1, 2, 3, 4$ in some permutation. What is $ABD_5$ in base 10?

*Solution.* The condition that $\{A, B, C, D, E\} = \{0, 1, 2, 3, 4\}$ forces each of these digits to be distinct. Clearly, we have that $D = C + 1$. Moreover, since the second digit changes between the second and third number, we must have carried a digit, which forces $E = 0$, causing $D = 4$ and $C = 3$. Furthermore, since the third digit remains invariant, we only carried the second digit and $A = B + 1$. This forces $A = 2$ and $B = 1$. Hence $ABD_5 = 214_5 = 2 \cdot 5^2 + 1 \cdot 5^1 + 4 \cdot 5^0 = \boxed{59_{10}}$.

*Solved by 82 out of 205 contestants* □

# 4 Team Round

**Problem 4.1** (Team problem 1)

Two coins are in a sack. One is a fair coin whose probability of coming up heads is $\frac{1}{2}$; the other coin is an unfair coin whose probability of coming up heads is $p$, where $p \in (0, 1)$ is a number to be determined later. One coin is selected from the sack at random and tossed. Given that it came up heads, what is the smallest (over all possible $p$) probability that the same coin will come up heads when tossed a second time?

*Solution.* Similar to the solution for Individual **14**, the probability that it is the biased coin that is selected is $\dfrac{p}{p+\frac{1}{2}} = \dfrac{2p}{2p+1}$ and the probability that it is the unbiased coin as $\dfrac{\frac{1}{2}}{p+\frac{1}{2}} = \dfrac{1}{2p+1}$. Thus, the probability of heads is $\dfrac{2p}{2p+1} \cdot p + \dfrac{1}{2p+1} \cdot \frac{1}{2}$. To minimize this, it suffices to check the value of this function at the critical points and the endpoints. Clearly, $p = 0$ or $p = 1$ produces a minimum value of $\dfrac{1}{2}$ and $\dfrac{5}{6}$. Now, take a derivative to find the critical points, getting that the derivative is $\dfrac{4p^2+4p-1}{(2p+1)^2}$. The critical points occur when the derivative is undefined or zero, and p is in the domain. The only critical point of the function is when $p = \dfrac{\sqrt{2}-1}{2}$, producing the minimum probability of $\boxed{\sqrt{2}-1}$.

*Solved by 7 out of 36 teams* □

**Problem 4.2** (Team problem 2)

Find the smallest integer $n$ so that $n > 1$ and the product of all of the positive integer divisors of $n$ is $n^7$.

*Solution.* The factors of any whole number come in pairs. If $A$ is a factor of $X$, then there must be a complementary number $B$ whose product with $A$ gives $X$. Therefore, if the product of all the factors of a certain $n$ is equal to $n^7$, then we know that $n$ has a total of 14 factors.

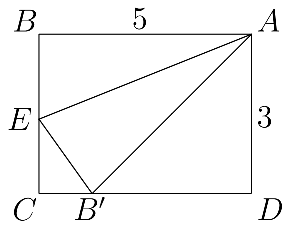
For a number $N$ whose prime factorization can be written as $a^A b^B c^C d^D e^E$..., the total number of factors can be expressed as a product $(A+1)(B+1)(C+1)$... This theorem can be best understood through combinatorics, with each distinct factor being a unique combination of the primes that compose $N$.

If $n$ has 14 factors, we now know that the prime factorization of $n$ must be either $a^{13}$ or $a^6 b^1$. For $n$ to be as small as possible, $n$ must have prime factorization of $a^6 b^1$, with $a$ and $b$ being 2 and 3 respectively. Thus, $2^6 * 3 = \boxed{192.}$

*Solved by 15 out of 36 teams* □

**Problem 4.3** (Team problem 3)

Rectangle $ABCD$ has $AB = 5$ and $AD = 3$. Fold the rectangle creased at $AE$ so that $B$ lands at $B'$ on $CD$. Find $BE$.




*Solution.* Since $AB'$ is just the reflection of $AB$ across $AE$, we have that $AB' = 5$ and hence $B'D = 4$ via the pythagorean theorem. Now since $\angle AB'D + \angle EB'C + 90 = 180$, we have that $\angle ECB = 90 - \angle EB'C = AB'D$, and because both triangles are right, they are similar by AA. Hence we have

$$\frac{EB'}{CB'} = \frac{EB'}{1} = \frac{AB'}{AD} = \frac{5}{3}.$$

Since $BE = EB'$, our final answer is $\boxed{\frac{5}{3}}$.

*Solved by 27 out of 36 teams* □

**Problem 4.4** (Team problem 4)

Find two positive integers $a$ and $b$ such that $a^2 + b^2 = 85113$ and the least common multiple of $a$ and $b$ is 1764. Express your answer as an ordered pair $(a, b)$.

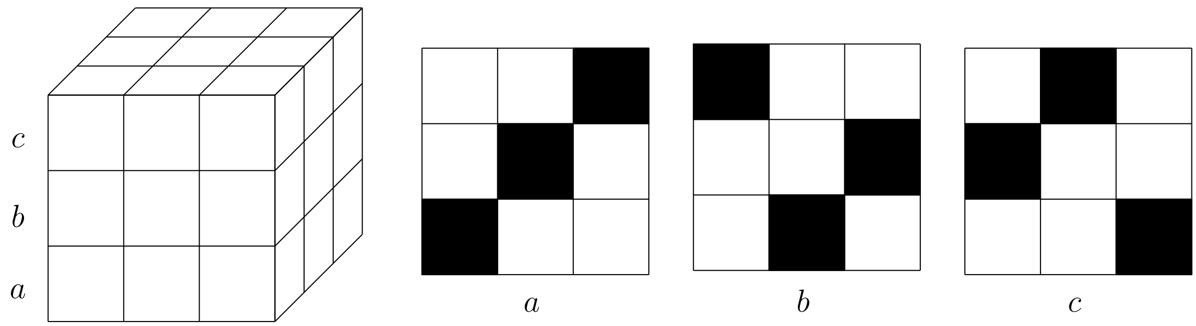
*Solution.* $1764 = 2^2 \cdot 3^2 \cdot 7^2$, $85113 = 3^2 \cdot 7^2 \cdot 193$. WLOG suppose $3|a$. Then $9|a^2$ and $9|85113$, so $9|(85113 - a^2) = b^2$ and $3|b$. Similarly, 7 divides both a and b. 441 divides each term in $a^2 + b^2 = 85113$. Let $21m = a$, $21n = b$. We have $m^2 + n^2 = 193$, and $\text{lcm}(m, n) = \frac{1764}{21} = 84$. Since m and n are integers, by trial and error we find the solution of $(m, n) = (7, 12)$ or $(12, 7)$, so $(a, b) = (147, 252)$ or $(252, 147)$.

*Solved by 16 out of 36 teams* □

**Problem 4.5** (Team problem 5)

We have 27 cubes, each $1 \times 1 \times 1$, of which some are transparent and some are opaque. We stack them together to make a $3 \times 3 \times 3$ cube. When projected onto the planes parallel to its faces, this appears as an opaque $3 \times 3$ square in each of the three directions. What is the smallest number of opaque cubes needed to do this?

*Solution.* Clearly we must have $\geq 9$ cubes. We can construct a winning example for 9 cubes as follows: split the cube into 3 layers oriented horizontally. We can color these layers separately and put them back together to form the whole. If we color them as below, we win, so $\boxed{9}$ is the minimum amount of opaque cubes necessary.




*Solved by 15 out of 36 teams*

□

**Problem 4.6** (Team problem 6)
Compute this sum:

$$\frac{2^{\frac{1}{1000}}}{2^{\frac{1}{1000}}+2^{\frac{1}{2}}}+\frac{2^{\frac{2}{1000}}}{2^{\frac{2}{1000}}+2^{\frac{1}{2}}}+\frac{2^{\frac{3}{1000}}}{2^{\frac{3}{1000}}+2^{\frac{1}{2}}}+\cdots+\frac{2^{\frac{999}{1000}}}{2^{\frac{999}{1000}}+2^{\frac{1}{2}}}$$

*Solution.* Note that

$$\frac{2^{\frac{k}{1000}}}{2^{\frac{k}{1000}}+2^{\frac{1}{2}}}+\frac{2^{\frac{1000-k}{1000}}}{2^{\frac{1000-k}{1000}}+2^{\frac{1}{2}}}=\frac{2^{\frac{k}{1000}}\cdot 2^{\frac{1000-k}{1000}}+2^{\frac{k+500}{1000}}+2^{\frac{1000-k}{1000}}\cdot 2^{\frac{k}{1000}}+2^{\frac{1500-k}{1000}}}{2^{\frac{k}{1000}}\cdot 2^{\frac{1000-k}{1000}}+2^{\frac{k+500}{1000}}+2^{\frac{1500-k}{1000}}+2^{\frac{1}{2}}\cdot 2^{\frac{1}{2}}}=1$$

We have 499 of these pairs, and a final $\frac{2^{\frac{500}{1000}}}{2^{\frac{500}{1000}}+2^{\frac{1}{2}}}=\frac{1}{2}$ so our final sum is $\boxed{\frac{999}{2}}$.

*Solved by 12 out of 36 teams*

□

**Problem 4.7** (Team problem 7)
What is the value of $\cos^2 10° + \cos^2 50° - \sin 40° \sin 80°$ ?

*Solution.* We can rewrite the expression using the identity $\sin(x) = \cos(90 - x)$, giving us that

$$\cos^2 10° + \cos^2 50° - \sin 40° \sin 80° = \cos^2 10° + \cos^2 50° - \cos 50° \cos 10°.$$

Now notice that this expression is in the form of the Law of Cosines on a triangle $ABC$ with $AB = \cos 10°$, $AC = \cos 50°$, and $\angle BAC = 60°$, as shown below:

Now since $60 = 50 + 10$, we can construct a point $P$ on the circumcircle of this triangle such that $\angle BAP = 10°$, $\angle CAP = 50°$, and $AP$ is a diameter of the circumcircle. Moreover, from the law of sines, we have that $BP = \sin 10°$ and $CP = \sin 50°$.

Now from the pythagorean identity we have that $AP^2 = \cos^2 50° + \sin^2 50° = 1$, so $AP = 2R = 1$, (where $R$ is the circumradius). Since we want $BC^2$, we solve for $BC$. From the extended law of sines, we have that

$$\begin{aligned} \frac{BC}{\sin 60°} &= 2R \\ BC &= 2R \sin 60° \\ &= (1)\left(\frac{\sqrt{3}}{2}\right) = \frac{\sqrt{3}}{2}, \end{aligned}$$

and our final answer is $BC^2 = \boxed{\frac{3}{4}}$.

*Solved by 6 out of 36 teams* □

**Problem 4.8** (Team problem 8)

Let $S$ be the set of all ordered pairs $(x, y)$ of *nonnegative* integers such that

$$x^3 + y^3 + 21xy = 343.$$

If $S = \{(x_i, y_i) : 1 \le i \le n\}$ is a set with $n$ members, find $\sum_{i=1}^{n} x_i + \sum_{i=1}^{n} y_i$.

*Solution.* Make the substitution $u = x + y$, $v = xy$. The diophantine then becomes $u(u^2 - 3v) + 21v = 343$, which we can factor as

$$\begin{aligned} 0 &= u^3 - 343 - 3uv + 21v \\ &= (u-7)(u^2 + 7u + 49) - 3v(u-7) \\ &= (u-7)(u^2 + 7u + 49 - 3v). \end{aligned}$$

Hence we now have two cases:
Case 1: $u - 7 = 0$. Any nonnegative integer solution to $x + y = 7$ satisfies this constraint, and since there are 8 such solution pairs, this case increments our sum by $7 \times 8 = 56$.
Case 2: $u^2 + 7u + 49 - 3v = 0$. Notice that since $x, y \ge 0$, we can invoke the AM-GM inequality, which tells us that

$$u^2 = (x+y)^2 \ge 4xy = 4v,$$

hence we have that for nonnegative $x, y$ $u^2 - 3v > u^2 - 4v \ge 0$ and thus this case never produces solutions. Therefore, our final sum is just $\boxed{56}$.

*Solved by 11 out of 36 teams* □