Math Day at the Beach 2016

Multiple Choice – *Write your name and school and mark your answers on the answer sheet. You have 30 minutes to work on these problems. No calculator is allowed.*

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1. What is the median of the following five values: $\sin 289°, \sin 145°, \sin 365°, \cos 72°, \frac{1}{2}$?

   (A) $\sin 289°$ (B) $\sin 145°$ (C) $\sin 365°$ (D) $\cos 72°$ (E) $\frac{1}{2}$

2. $\sin(\frac{\pi}{6} + \arcsin(x)) = ?$

   (A) $\frac{1}{2} + x$ (B) $\frac{\sqrt{3}}{2} + x$ (C) $\frac{\sqrt{3}}{2}\sqrt{1-x^2} + \frac{1}{2}\,x$ (D) $\frac{1}{2}\sqrt{1-x^2} + \frac{\sqrt{3}}{2}\,x$ (E) $\frac{\sqrt{3}}{2}\sqrt{1-x^2} - \frac{1}{2}\,x$

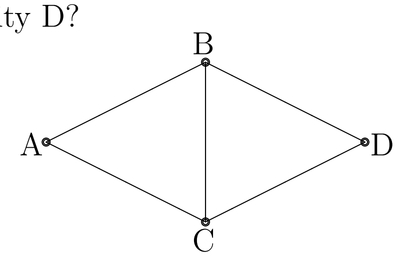
3. Choose a string of 10 digits in such a way that each digit is chosen independently with each of $\{0, 1, \ldots, 9\}$ being equally likely. Let P be the probability that each digit appears in this string exactly once. (For instance, 0124356987.) Let $L = \log_{10} P$. Which of the following is true?

   (A) $L \le -5$ (B) $-5 < L \le -4$ (C) $-4 < L \le -3$ (D) $-3 < L \le -2$ (E) $-2 < L \le 0$

4. f is a quadratic polynomial of the form $f(x) = ax^2 + bx + c$ such that $f(1) = 0$, $f(3) = 0$, and $f(-1) = 24$. If $N = f(225)$, find the remainder left when N is divided by 5.

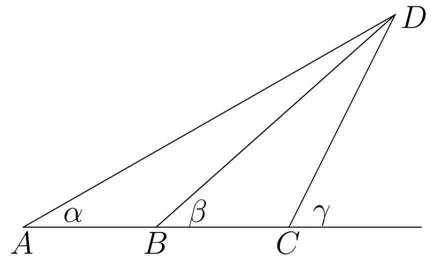
   (A) 0 (B) 1 (C) 2 (D) 3 (E) 4

5. There are four cities linked by roads as in the picture. Each segment of road linking adjacent cities has, independently, a probability $\frac{1}{2}$ of being shut down. What is the probability that one can travel from city A to city D?



   (A) $\frac{3}{8}$ (B) $\frac{1}{2}$ (C) $\frac{5}{8}$ (D) $\frac{3}{4}$ (E) $\frac{7}{8}$

6. If $AB = BC$ and all of the lines shown meet at D, which of the following must be an arithmetic progression?



(A) $\sin\alpha, \sin\beta, \sin\gamma$ (B) $\cos\alpha, \cos\beta, \cos\gamma$ (C) $\tan\alpha, \tan\beta, \tan\gamma$ (D) $\cot\alpha, \cot\beta, \cot\gamma$
(E) α, β, γ

7. Calculate the area in the plane of the set of all points (x, y) such that $|x-1| + 2|y| \le 1$.

(A) $\frac{1}{4}$ (B) 1 (C) 2 (D) 4 (E) The area is infinite.

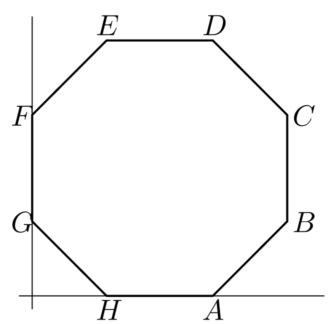
8. At a certain meeting, there are twice as many women present as men. $\dfrac{3}{4}$ of the men are wearing some item of blue clothing. If half of all the people present are wearing blue, what is the percentage of the women wearing blue?

(A) 0% (B) 25% (C) 37.5% (D) $\approxeq 41.7\%$ (E) 50%

9. How many integers n are there with $2 \le n \le 100$ such that the smallest prime factor of n is greater or equal to 7?

(A) 21 (B) 22 (C) 23 (D) 24 (E) 25

10. Assume a and b are real numbers, not both zero. Suppose that the function $ax + by + 5$ has its maximum value on the region enclosed by the regular octagon shown below at the point B and nowhere else. Find the vertex at which $5 - bx - ay$ has its maximum value.



(A) A (B) B (C) C (D) D (E) E

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Individual Free Response Part 2 – *Write your name and school and mark your answers on your answer sheet. You have 20 minutes to work on these problems. No calculator is allowed.*

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11. If $2015 = b^2 - a^2$, for positive integers b and a, what is the smallest possible value for b? (Hint: $2015 = 5 \cdot 13 \cdot 31$.)

12. Let S be the set of all positive integers n such that $\dfrac{n^2 - 20}{n + 14}$ is a positive integer. Find the sum of the elements of S.

13. Two square pyramids with the same altitude are placed in space so that they intersect. Their bases lie in parallel planes, corresponding edges of their bases are parallel, and each pyramid's vertex lies at the center of the base of the other pyramid. Each pyramid has altitude 8. One pyramid has a base of side 6 and the other has a base of side 10. Compute the volume of the intersection of the two pyramids.

14. An 8×8 checkerboard can be folded into 4×8 halves. On each half of the checkerboard, three small Velcro squares are glued randomly onto three board squares. (That is, three squares on one half and three more, complementary, squares on the other half.) Fold the board. Let P be the probability that the board sticks shut. (That only happens if there is Velcro glued to corresponding squares on each side.) If $P = \dfrac{a}{b}$, where a and b are integers and the fraction is in lowest terms, compute $a + b$.

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INDIVIDUAL FREE RESPONSE PART 3 – *Write your name and school and mark your answers on your answer sheet. You have 20 minutes to work on these problems. No calculator is allowed.*

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15. If the point (a, b) is reflected in the line $x + y = 1$, find the coordinates of the reflected point.

16. Let S be the set of complex numbers of the form $z = x + iy$ such that x and y are integers and $x = y^2 - 1$. Let N be the product of the 11 members of S that lie closest to the origin. Find the number of positive integer factors of $|N|$.

17. A robot is standing at an integer point on a number line. The robot flips two fair coins. If either or both come up heads, it take one step to the right, to the next larger integer. If both come up tails, it takes one step to the left, to the next smaller integer. If it ever reaches the point 0, the robot stops and shuts down. If the robot starts at point 2, what is the probability that it ever stops?

18. Let $f(n) = \dfrac{4n + \sqrt{4n^2 - 1}}{\sqrt{2n+1} + \sqrt{2n-1}}$. Compute $f(1) + f(2) + f(3) + \cdots + f(12)$.

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Team Round – *Write your school name and mark your answers on the answer sheet. No calculator is allowed. You have 30 minutes to work on these problems.*

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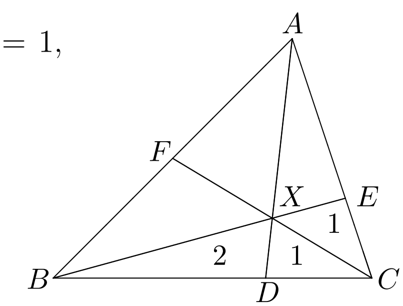
1. In how many ways can you write 1025 as a sum of nonnegative integer powers of 2 so that each power of 2 appears at most twice?

2. The great circle distance between city A and city B is $\frac{1}{6}$ of the circumference of the earth. A satellite in orbit can "see" both cities at the same time. If h is the smallest possible altitude above the earth's surface for the satellite, compute $\frac{h}{R}$, where R is the radius of the earth. (Assume the earth is a sphere.)

3. There are 2016 slips of paper with numbers written on them, numbered from 1 through 2016, laid out in order around the edge of a large round table. Sasha starts walking around the table picking up every other slip of paper she sees. She picks up #1, skips over #2, picks up #3, skips over #4, picks up #5, and so on. She keeps walking around and around the table until she has picked up every slip of paper. What is the number on the very last slip of paper that she picks up?

4. In the figure to the right, area($\triangle BDX$) = 2, area($\triangle DCX$) = 1, and area($\triangle CEX$) = 1. Compute area($\triangle ABC$).



5. Find the product of the slopes of the two tangent lines from the point $(0, \sqrt{2})$ to the ellipse $\frac{x^2}{9} + y^2 = 1$.

6. Suppose x, y, z are real numbers such that $x + y + z = 21$ and $xy + xz + yz = 99$. Find the maximum possible value of xyz.

7. Let $x_1 = \frac{355}{113}$, and after that, $x_{n+1} = \frac{x_n^2}{2} + x_n$. Compute $\sum_{n=1}^{\infty} \frac{1}{2 + x_n}$.

8. Inside a square of area 1, an isosceles triangle is inscribed with its unique vertex at a corner of the square, and then a smaller square is inscribed in the triangle, as shown. Find the largest possible area for the smaller square.