Math Day at the Beach 2015

Multiple Choice – *Write your name and school and mark your answers on the answer sheet. You have 45 minutes to work on these problems. No calculator is allowed.*

---

1. Let $N = \lfloor 100\pi \rfloor$ (That is, the greatest integer that is less than 100π.) Find the sum of all of the positive integer divisors of N.

   (A) 2 (B) 4 (C) 314 (D) 474 (E) 628

2. The 10th root of the number $10^{10^{10}}$ (which means $10^{(10^{10})}$) is

   (A) 10^{10} (B) 10^{10^9} (C) $10^{10^{10}-10}$ (D) $\left(\sqrt{10}\right)^{10^{10}}$ (E) $10^{10^{10}-1}$

3. Which of the following lines divides the circle $x^2 + 4x + y^2 + 6y = 10$ into two equal parts?

   (A) $y = 4x - 5$ (B) $y = 5x + 7$ (C) $y = 6x - 9$ (D) $y = 7x - 19$ (E) $y = 8x + 10$

4. Two circles of radii 3 inches and 9 inches are externally tangent. $\overline{AB}$ is the common external tangent. Find the length AB in inches,

   (A) $6\sqrt{3}$ (B) 10 (C) $6\sqrt{5}$ (D) $3\sqrt{5}$ (E) 12

5. How many *distinct* numbers are there in the following collection?

   $$\left\{ \ln(4-\sqrt{15}),\ \ln(4+\sqrt{15}),\ -\ln(4-\sqrt{15}),\ -\ln(4+\sqrt{15}),\ \ln\left(\frac{4+\sqrt{15}}{4-\sqrt{15}}\right),\ \ln(31+8\sqrt{15}) \right\}$$

   (A) 2 (B) 3 (C) 4 (D) 5 (E) 6

6. The vertices of a triangle in the coordinate plane are $(0,4), (1,5)$, and $(2,3)$. Find its area.

   (A) $\frac{3}{2}$ (B) 2 (C) $\frac{5}{2}$ (D) 3 (E) $\frac{7}{2}$

7. You play the following game with a friend. You share a pile of chips, and you take turns removing between one and four chips from the pile. (In particular, at least one chip must be removed on each turn.) The game ends when the last chip is removed from the pile; the one who removes it is the loser.

   It is your turn, and there are 2015 chips in the pile. How many chips should you remove to guarantee that you win, assuming you then make the best moves until the game is over?

   (A) 1 (B) 2 (C) 3 (D) 4 (E) There is no way to guarantee a win even with best play.

8. Compute $\tan\left(3\arctan\left(\frac{1}{3}\right)\right)$.

   (A) 1 (B) $\frac{13}{3}$ (C) $\frac{13}{9}$ (D) $\frac{27}{26}$ (E) undefined

9. Three distinct numbers are chosen at random from the primes that are less than 20. What is the probability that they are the sides of a nondegenerate triangle?

(A) $\frac{1}{4}$ (B) $\frac{15}{56}$ (C) $\frac{2}{7}$ (D) $\frac{9}{28}$ (E) $\frac{1}{2}$

10. The faces of a closed polyhedron consist of some hexagons and some quadrilaterals. Exactly 3 faces meet at each vertex. Find the smallest possible number of quadrilaterals.

(A) 0 (B) 1 (C) 2 (D) 4 (E) 6

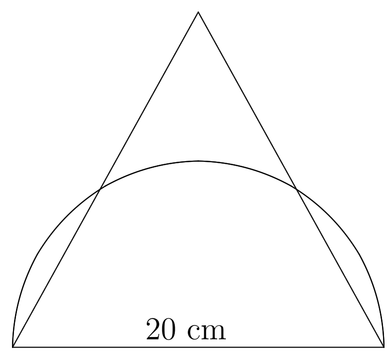
11. Rhombus $ABCD$ has side length x. Point O is placed on diagonal $\overline{AC}$ such that $OA = x$ and $OB = OC = OD = 1$. Compute x.

(A) $\sqrt{3}$ (B) $\frac{1+\sqrt{3}}{2}$ (C) 1 (D) $\frac{\sqrt{5}-1}{2}$ (E) $\frac{\sqrt{5}+1}{2}$

12. Professor Mena will be on sabbatical for 255 days. He has asked Professor Gao to water his flowers while he is gone. Professor Gao must water the daisies every 3rd day, the petunias every 5th day, and the begonias every 17th day. How many days will Professor Gao not have to water any flowers?

(A) 103 (B) 104 (C) 119 (D) 128 (E) 129

13. As shown below, an equilateral triangle with edge length 20 cm and a semicircle share an edge. If the region that lies outside the triangle but inside the semicircle has area μ cm^2, then



(A) $0 < \mu < 6$ (B) $6 < \mu < 10$ (C) $10 < \mu < 11$ (D) $11 < \mu < 16$ (E) $\mu > 16$

14. Let $f(x) = \dfrac{x}{\sqrt{1+x^2}}$, and let $f_n(x) = \underbrace{f(f(f(\cdots(f}_{n}(x))\cdots))))$. That is, $f_1(x) = f(x)$ and we recursively define $f_{n+1}(x)$ as $f(f_n(x))$. Find the smallest positive n such that $f_n(2) \leq \frac{2}{17}$.

(A) 17 (B) 36 (C) 72 (D) 289 (E) Never (that is, $f_n(x) > \frac{2}{17}$ for all n.)

15. Compute the product $\cos\left(\frac{\pi}{5}\right)\cos\left(\frac{2\pi}{5}\right)\cos\left(\frac{3\pi}{5}\right)\cos\left(\frac{4\pi}{5}\right)$.

(A) $\frac{1}{16}$ (B) $\frac{\sqrt{5}}{32}$ (C) $\frac{\sqrt{5}-1}{16}$ (D) $\frac{5+\sqrt{5}}{32}$ (E) $\frac{5}{16}$.

Math Day at the Beach 2015

Individual Free Response – *Write your name and school and mark your answers on your answer sheet. You have 25 minutes to work on these problems. No calculator is allowed.*

---

16. The parabolas $y = 3x^2$ and $x = 24y^2$ intersect at two points. Find the slope of the line through those two points.

17. Lay six long strings in parallel. Two people (each of whom cannot see what the other is doing) work at opposite ends of the bundle of strings. Each randomly pairs off the strings at his or her end and glues together the ends of each pair. Then stretch the strings out. What is the probability that the strings now form a single large loop?

18. Find the minimum value of y if $y = |x-1| + |x-2| + |x-3| + \cdots + |x-2015|$ for any real x.

19. Three solid spheres with radii 1, 2, and 3 are all tangent to a plane, and each sphere is also externally tangent to the other two spheres. The three points at which each sphere is tangent to the plane are the vertices of a triangle in that plane. Compute the area of the triangle.

20. If α, β are the two roots of $x^2 + \sqrt{10}x + 2 = 0$, find $\left| \dfrac{\alpha^4 + \alpha^2\beta^2 + \beta^4}{\alpha^2 - \beta^2} \right|$.

Math Day at the Beach 2015

TEAM ROUND – *Write your school name and mark your answers on the answer sheet. No calculator is allowed. You have 30 minutes to work on these problems.*

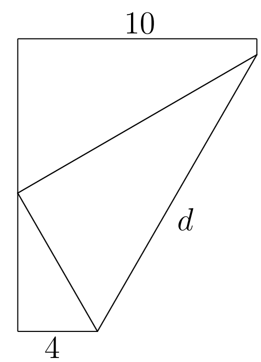
---

1. How many integers are there between 22 and 4444 such that their base 5 representations have no differing digits, as in 33_5 or 1111_5? (Note that the numbers 22 and 4444 in this problem are expressed in base 10 notation.)

2. Find the shortest distance from the point $(-26, 0)$ to the curve $\sqrt{x} + \sqrt{y} = 4$.

3. Toss a fair coin 10 times. What is the probability that at some point in this string of tosses, heads appears at least 5 times in a row?

4. Take a piece of paper that is 10 inches by 15 inches. Fold it as shown, starting the crease a distance 4 inches along the short side from one corner, and make the folded corner touch the edge. Find the length (in inches) of the crease (labeled d in the figure).



5. What is the area of the triangle bounded by the x-axis, the y-axis, and the line tangent to the curve $x = 3y^2$ at the point $(\frac{1}{3}, \frac{1}{3})$.

6. Let S be the unit sphere. (That is, radius 1, centered at the origin in $\mathbb{R}^3$.) For two points P, Q on S the distance $d(P, Q)$ is defined to be the (shorter) great circle arc length connecting P and Q. Find $d(P, Q)$ where $P = \frac{1}{4}(-\sqrt{2}, -\sqrt{2}, 2\sqrt{3})$ and $Q = \frac{1}{4}(\sqrt{3} - 1, \sqrt{3} - 1, \sqrt{2} + \sqrt{6})$.

7. Find all positive integers n such that $n^2 + 999$ and $n^2 + 2000$ are both perfect squares.

8. Any integer k can be uniquely represented as $k = \sum_{j=0}^{m} d_j \cdot 3^j$ where $d_j \in \{-1, 0, 1\}$. Let $f(k) = \sum_{j=0}^{m} d_j$ for that representation. For instance, $f(3) = 1$ since $3 = 0 \cdot 3^0 + 1 \cdot 3^1$ and $f(5) = -1$ since $5 = (-1) \cdot 3^0 + (-1) \cdot 3^1 + 1 \cdot 3^2$. Next, let $g(n) = \sum_{k=1}^{n} f(k)$. For instance, $g(3) = f(1) + f(2) + f(3) = 1 + 0 + 1 = 2$.

   Count how many numbers n there are with $1 \leq n \leq 2015$ such that $g(n) = n$.