Do three problems from each part, for a total of six problems.

PART A

1. Use appropriate theorems or techniques of complex analysis to show that

\[ \int_{0}^{\infty} \frac{\sin x}{x} \, dx = \frac{\pi}{2} \]

2. (a) State Rouche’s Theorem.
   (b) Consider the equation \( z^5 + 8z + 3 = 0 \).
       How many roots of this equation are inside the open unit disk?
   (c) How many roots of this same equation are inside the annulus
       \( A = \{ z : 1 < |z| < 2 \} \)?

3. Find the Laurent series for the function \( f(z) = \frac{e^{2z} - 1}{z^4} \) that is valid on a punctured neighborhood of zero. Determine on what set this series converges and find the residue of \( f \) at zero.

4. Compute \( \int_{-\infty}^{\infty} \frac{\cos x}{x^4 + 5x^2 + 4} \, dx. \)
   Hint: first show that it is equal to \( \int_{-\infty}^{\infty} \frac{e^{ix}}{x^4 + 5x^2 + 4} \, dx. \)

5. (a) State the Cauchy-Riemann equations.
   (b) Suppose \( u(x, y) = xe^x \cos y - ye^x \sin y \). Find \( v \) such that \( u + iv \) is analytic.
6. A Möbius transformation \( \varphi : \mathbb{C}_\infty \to \mathbb{C}_\infty \) is a function of the form \( \varphi(z) = \frac{az+b}{cz+d} \), where \( a, b, c, d, \in \mathbb{C} \) and \( ad - bc \neq 0 \). Let \( D = \{ z : |z| < 1 \} \) be the open unit disk.

(a) Let \( \varphi(z) = \frac{1}{z-1} \). Describe \( \varphi(D) \), the image of \( D \) under \( \varphi \). Justify your answer.

(b) Find a Möbius transformation \( \varphi \) satisfying \( \varphi(D) = D \) and \( \varphi\left(\frac{1}{2}\right) = \frac{3}{4} i \).

7. (a) Let \( f \) be analytic on a nonempty region \( G \). Prove that if \( |f(z)| = 1 \) for all \( z \in G \), then \( f \) is a constant.

(b) Let \( f \) be an entire function. Prove that if \( |f(z)| \leq e^{\operatorname{Re} z} \) for all \( z \in \mathbb{C} \), then there exists a constant \( |c| \leq 1 \) such that \( f(z) = ce^{z} \) for all \( z \in \mathbb{C} \).

8. (a) State Morera’s Theorem.

(b) Let \( \{ f_n \} \) be a sequence of analytic functions on a region \( G \). Let \( f \) be a continuous function on \( G \). Prove that if \( f_n \to f \) uniformly on every compact subset of \( G \), then \( f \) is analytic on \( G \).

9. Suppose \( g(z) = \frac{P(z)}{Q(z)} \) is a rational function (a polynomial divided by a polynomial) such that the degree of the denominator is at least two greater than the degree of the numerator. Prove that sum over all poles of the residues of \( g \) is zero. (Hint: use an integral over a large circle.)

10. Suppose that \( f(z) \) is analytic on \( H = \{ z : \operatorname{Re} z > 0 \} \) and that \( |f(z)| \leq 1 \) for all \( z \in H \). Prove that \( |f'(z)| \leq \frac{1}{\operatorname{Re} z} \) for all \( z \in H \).