CALIFORNIA STATE UNIVERSITY, LONG BEACH
DEPARTMENT OF MATHEMATICS AND STATISTICS

SPRING 2006
COMPLEX ANALYSIS COMPREHENSIVE EXAMINATION
FEBRUARY 18, 2006

Do three problems from each part, for a total of six problems.

PART A

1. Calculate the integral \[ \int_{-\infty}^{\infty} \frac{x^2}{x^4 + 1} \, dx \]

2. Find the Laurent series of \( f(z) = \frac{z-3}{(z-1)(z-2)} \) with center 0 that is valid in each of the following regions
   a. \( |z| < 1 \)
   b. \( 1 < |z| < 2 \)
   c. \( 2 < |z| \)

3. Consider the family of lines \( L_k = \{ x + ki : x \in \mathbb{R} \} \), where \( k \) ranges over the positive integers. Assuming that \( f(z) = \frac{3z-4}{z-1} \) maps the upper half plane onto itself, sketch a graph showing what the collection of images \( f(L_k) \) look like.

4. For each of the following functions:
   i. Identify the singularity at the given point as removable, a pole (state the order), or essential.
   ii. Calculate the residue at the given point
      a. \( \tan z \) at \( z = \frac{\pi}{2} \)
      b. \( \frac{\sin(3z)}{z^6} \) at \( z = 0 \)
      c. \( e^{\cot(z)} \) at \( z = 0 \)

5. a. State Rouche’s theorem.
   b. Counting multiplicities, how many zeros does the function \( f(z) = 6z^4 - 2z + e^z \) have in the disk \( \{ z : |z| < 1 \} \)? Prove your answer.
PART B

1. Let $f$ be an entire function and suppose there is a constant $M$, an $R > 0$, and an integer $n \geq 1$ such that $|f(z)| \leq M |z|^n$ for $|z| > R$. Show that $f$ is a polynomial of degree $\leq n$.

2. Let $U : \mathbb{C} \rightarrow \mathbb{C}$ be a harmonic function such that $U(z) \geq 0$ for all $z$ in $\mathbb{C}$, prove that $U$ is constant.

3. a. State Schwarz lemma.
b. Let $D$ denote the unit disk and suppose that $f : D \rightarrow D$ is analytic. Show that if $f(a) = a$ and $f(b) = b$ for two distinct complex numbers $a$ and $b$ in $D$ then $f(z) = z$. Hint: Apply Schwarz lemma to the function $\phi \circ f \circ \phi^{-1}$ where $\phi(z) = \frac{z-a}{1-\bar{a}z}$.

4. Suppose $f : \mathbb{C} \rightarrow \mathbb{C}$ is a polynomial and that $f'(z)$ is never equal to 0. Show that $f$ must be one-to-one on $\mathbb{C}$. Is this true for an arbitrary entire function? (Prove or give a counterexample.)

5. a. State the Fundamental Theorem of Algebra.
b. Give two different proofs of the Fundamental Theorem of Algebra.