Choose six problems total, including at least two problems from Part I and two problems from Part II. Please indicate clearly which problems you want graded.

**Part I: Groups** (Choose at least two.)

1. Let $G$ be a group. For $g \in G$, let $|g|$ be the order of $g$. Suppose $x, y \in G$ with $|x| = 2$ and $|y| = 3$.
   (a) Prove that if $x$ and $y$ commute, then $|xy| = 6$.
   (b) Give an example of $G$, $x$, and $y$ that satisfy the initially stated conditions (but $x$ and $y$ do not commute) such that $|xy| = 3$.
   (c) Give an example of $G$, $x$, and $y$ that satisfy the initially stated conditions (but $x$ and $y$ do not commute) such that $|xy| = 4$.

2. (a) Find the conjugacy classes in $A_4$. Use these to find all proper normal subgroups of $A_4$.
   (b) Show that $A_4$ is the only subgroup of $S_4$ of order 12.
   (c) Let $G$ be a nonabelian finite group with center $Z(G)$. Show that if $[G : Z(G)] = n$, then every conjugacy class of $G$ has strictly fewer than $n$ elements.

3. Let $G$ be a group. Define the commutator subgroup of $G$ to be the subgroup $G'$ generated by all elements of the form $a^{-1}b^{-1}ab$, where $a, b \in G$.
   (a) Prove each of the following statements:
      i. $G'$ is a normal subgroup of $G$.
      ii. The quotient group $G/G'$ is abelian.
      iii. If $f : G \to H$ is a group homomorphism and $H$ is abelian, then $G' \subseteq \ker f$.
   (b) Give an explicit description of $G'$ for $G = D_8$, the dihedral group of order 8.

4. (a) Prove that any group of order 105 has a subgroup of order 35.
   (b) Describe all isomorphism classes of abelian groups of order 600.

5. For a group $G$, let $\text{Aut}(G)$ be the group of automorphisms of $G$ and let $Z(G)$ be the center of $G$.
   (a) Prove that $G/Z(G)$ is isomorphic to a subgroup of $\text{Aut}(G)$.
   (b) For $G = \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, determine $\text{Aut}(G)$. (In other words, give a well-known group to which $\text{Aut}(G)$ is isomorphic.)
6. Let $R$ be an integral domain. Label each of the following statements as true or false. Justify each answer with a proof or counterexample.

(a) For any two nonzero proper ideals $I, J$ of $R$, $I \cap J \neq \{0\}$.
(b) Every nonzero prime ideal of $R$ is maximal.
(c) Every prime element of $R$ is irreducible.
(d) If $R$ is finite, then $R$ is a field.

7. Consider the Gaussian integers $\mathbb{Z}[i]$, where $i = \sqrt{-1}$. For $z = a + bi \in \mathbb{Z}[i]$, we define $N(z) = a^2 + b^2$.

(a) Show that for $z_1, z_2 \in \mathbb{Z}[i]$, $N(z_1 z_2) = N(z_1)N(z_2)$.

(b) Let $z_1, z_2 \in \mathbb{Z}[i]$ with $z_2 \neq 0$. Show that there exist $q, r \in \mathbb{Z}[i]$ with $z_1 = qz_2 + r$ and $N(r) < N(z_2)$.

(c) For $z \in \mathbb{Z}[i]$, prove that if $N(z)$ is prime, then $z$ is irreducible.

(d) Give the factorization of $3 + i$ into irreducibles.

8. (a) Prove that every Euclidean domain is a principal ideal domain.

(b) A monic polynomial is one whose leading (highest degree) coefficient is $1$. Determine a monic polynomial $p(x) \in \mathbb{R}[x]$ that generates the ideal generated by $x^{12} - 1$ and $x^8 - 1$.

(c) Find polynomials $a(x), b(x) \in \mathbb{R}[x]$ such that

$$a(x)(x^{12} - 1) + b(x)(x^8 - 1) = p(x).$$

(d) Prove that $\mathbb{R}[x, y]$ is not a principal ideal domain.

9. Let $u = (1, 0, 2) \in \mathbb{R}^3$.

(a) Find a $3 \times 3$ matrix $M$ with the following properties:

i. $M$ has an eigenvalue 5 whose corresponding eigenspace is the set of vectors perpendicular to $u$.

ii. $u$ is in the nullspace of $M$.

(b) Compute the rank and nullity of $M$. 