Student Learning Outcomes
Undergraduate Courses for the Programs

In this appendix we include descriptors for what are considered some of the most crucial courses for the Majors in the Department.

The list of courses is

	General Option	Applied Math	Math Education	Statistics
Math 122	☑	☑	☑	☑
Math 123	☑	☑	☑	☑
Math 233	☑	✗	☑	✗
Math 247	☑	☑	☑	☑
Math 361A	☑	☑	✓	☑
Math 364A	✓	☑	✓	✗
Math 380	☑	☑	☑	☑
Math 381	✗	✗	✓	☑
Math 410	✗	✗	☑	✗
Math 444	☑	✗	✓	✗
Math 470	✗	☑	✗	✗

☑ Required and considered essential
✓ Required
✗ Not Required

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Major Goals:

The main goal of this course is to prepare students for higher mathematics. This is done by engaging students in deep problem-solving situations and techniques of proof that presage higher level topics. Through example and exercise, students will raise their general mathematical sophistication—the ability to read and write complex and convincing arguments. The mathematical reasoning in this course is practiced on fundamental topics that are foundational for higher mathematics. These topics include numbers, sets, induction, relations, functions, and counting techniques. The specific goals are as follows. Students will demonstrate the ability to

❶ Transform intuition into proof, and to differentiate between proof and opinion/example.

❷ Use the propositional and predicate logic and the language of sets, relations, and functions in writing mathematical proofs.

❸ Read and construct valid mathematical arguments (proofs), including proofs by induction, direct and indirect reasoning, proof by contradiction, and disproof by counterexample.

❹ Solve counting problems by applying the multiplication principle, the inclusion-exclusion principle, the pigeonhole principle, recurrence relations, and, in particular; the use of permutations and combinations.

❺ Use counting techniques to compute probabilities of events.

Representative Textbooks
A transition to advanced mathematics, by D. Smith, M. Eggen and R. St. Andre
A concise introduction to pure mathematics, by M. Liebeck

Assessment Method: Embedded questions throughout homework, quizzes and exams.

Illustration #1: Counting

Putting "Balls-into-Buckets" is a model for the following mathematical constructs.

1. Partitions: Modeled by distinguishable balls into identical buckets
2. Functions: Modeled by distinguishable balls into distinguishable buckets.
    a. Injective functions: at most one ball in each bucket
    b. Surjective functions: no empty buckets
    c. Bijective functions: exactly one ball in each bucket
3. The number of nonnegative solutions of a Diophantine equation can be modeled by identical balls into distinguishable buckets. For example, the number of solutions of $x+y+z=10$ is modeled by ten identical balls into three distinguished buckets.

Illustration #2: Relations

1. Define the relation on the set $1,-1,2,-2,3,-3$ by $a < b$ if a divides b. Is this relation a partial order? Prove your answer.
2. Define $a \approx b$ if ab is a square number. Is this relation an equivalence relation? Prove your answer.

Illustration #3: Functions

1. If $f \circ g$ is injective, prove that g is injective.
2. Give an example where $f \circ g$ is injective but f is not injective.

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Goals:

❶ Students will understand matrices as collectors of information, execute key operations involving matrices, and use these operations to solve relevant problems.

❷ Students will be able to find all solutions to arbitrary systems of linear equations, by using Gaussian Elimination. The student will understand the process behind the reduction, and eventually understand the relation between solvability, null spaces and column spaces.

❸ Students will increase their level of mathematical linguistic ability by being able to deduce simple truths or give counterexamples to falsities involving the many concepts introduced in the course such as linear independence, spanning sets, dimension and basis.

❹ Students will increase their sense of mathematical rigor by being able to argue simple theorems concerning matrices and vectors as well as follow more sophisticated arguments.

❺ Students will be able to compute eigenvalues and eigenvectors of a matrix, and their relation to whether a matrix can be similar to a diagonal matrix or not.

❻ Students will be able to orthogonally diagonalize any symmetric matrix by using the Gram-Schmidt process to find an orthonormal basis for such a matrix.

Assessment Method: Embedded questions throughout homework, quizzes and exams.

Illustration #1: Of the three options on the right in the table, check **all** those that are applicable as to the possible number of solutions to the system $\mathbf{Ax}=\mathbf{b}$, assuming all the information that you are given. For some, only one option may apply, for others, two may apply, and still others all three may apply. For example, if you are told that $\mathbf{A}$ is 3×3, and that it has rank 3, then you should put a check only on the Unique column, since such a system will have a unique solution. On the other hand, if all you are given is that $\mathbf{A}$ is 3×3, then you should put a check in all three columns No solution, Unique Solution, and Infinitely many since all of these are possible under the assumptions.

#	$\mathbf{Ax}=\mathbf{b}$	No Solution	Unique Solution	Infinitely many solutions
1	$\mathbf{A}$ is 12×15 and $\mathbf{b}=\mathbf{0}$			
2	$\mathbf{A}$ is 12×15			
3	$\mathbf{A}$ is 15×10			
4	$\mathbf{A}$ is 12×12			
5	$\mathbf{A}$ is 15×10 and $\mathbf{b}=\mathbf{0}$			

6	$\mathbf{A}$ is 12×12 and $\mathbf{A}$ has rank as large as possible			
7	$\mathbf{A}$ is 12×15 and $\mathbf{A}$ has rank as large as possible			
8	$\mathbf{A}$ is 15×10 and $\mathbf{A}$ has rank as large as possible			
9	$\mathbf{A}$ is 15×10 and both $\mathbf{A}$ and $\mathbf{A}\quad\mathbf{b}$ have rank as large as possible			
10	$\mathbf{A}$ is 12×15 and both $\mathbf{A}$ and $\mathbf{A}\quad\mathbf{b}$ have rank as large as possible			

The student is expected to fill the table correctly ❷, ❸, ❹.

In a similar fashion the student is expected to answer the following questions:

Illustration #2: Consider the veracity or falsehood of each of the following statements, and argue for those that you believe are true while providing a counterexample for those that you believe are false. Let $S = \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4$, $\mathbf{A} = \mathbf{u}_1 \quad \mathbf{u}_2 \quad \mathbf{u}_3 \quad \mathbf{u}_4$, $T = S \cup \mathbf{v} = \mathbf{v}, \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4$ ❸, ❹.

① $\mathbf{v}$ is in the span of T.

② $\mathbf{0}$ is in the span of T and in the span of S.

③ $\mathbf{Ax} = \mathbf{v}$ has a solution if and only if the span of S is the same as the span of T.

④ The rank of $\mathbf{A}$ is 4 if and only if S is a linearly independent set.

⑤ If T is linearly independent, then so is S.

Illustration #3 Show that the collection of symmetric matrices of size n is a subspace of the space of all square matrices of size n, and compute its dimension.

The proof should follow from elementary properties of the transpose. That $(a\mathbf{A} + \mathbf{B})^{\mathrm{T}} = a\mathbf{A}^{\mathrm{T}} + \mathbf{B}^{\mathrm{T}}$ for any scalar a and any matrices $\mathbf{A}, \mathbf{B}$ implies that the collection of symmetric matrices ($\mathbf{A}^{\mathrm{T}} = \mathbf{A}$) is a subspace ❸, ❹. To compute the dimension, one needs to find a basis. The student could start by doing the 2×2 case and obtain easily the basis $\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}$, $\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$ and $\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}$. So it is 3-dimensional. For the case $n=3$, an arbitrary symmetric case is $\begin{pmatrix}a & b & c\\ b & d & e\\ c & e & f\end{pmatrix}$, so one can see that a basis will consist of 6 elements: $\begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix}$, $\begin{pmatrix}0&1&0\\1&0&0\\0&0&0\end{pmatrix}$, $\begin{pmatrix}0&0&1\\0&0&0\\1&0&0\end{pmatrix}$, $\begin{pmatrix}0&0&0\\0&1&0\\0&0&0\end{pmatrix}$, $\begin{pmatrix}0&0&0\\0&0&1\\0&1&0\end{pmatrix}$ and $\begin{pmatrix}0&0&0\\0&0&0\\0&0&1\end{pmatrix}$. From there the student is expected to conclude that the basis will always have for arbitrary n, $\frac{n \ \ n+1}{2}$ elements so the dimension given by this number ❹.

Illustration #4: Suppose $\mathbf{A}$ is 10×10. Suppose $\mathbf{A}-\mathbf{I}$ has rank 3, $\det \ \mathbf{A}+\mathbf{I} \ =0$ and $\det \ \mathbf{A}-2\mathbf{I} \ =0$. Suppose the trace of $\mathbf{A}$ is 9. Decide whether $\mathbf{A}$ is invertible and whether $\mathbf{A}$ is similar to a diagonal matrix.

Here the student is expected to deduce that 1 is an eigenvalue with multiplicity at least 7 ❸, ❺, and that also -1 and 2 are eigenvalues. Finally since the trace is 9, the last remaining eigenvalue is also 1. From there the student concludes that $\mathbf{A}$, while invertible, is not similar to a diagonal matrix ❸, ❹, ❺.

Illustration #5 Let $\mathbf{A}=\mathbf{J}_5=\begin{pmatrix}1&1&1&1&1\\1&1&1&1&1\\1&1&1&1&1\\1&1&1&1&1\\1&1&1&1&1\end{pmatrix}$. To find an orthogonal matrix whose columns are eigenvectors, first one must compute the eigenvalues to be $0,0,0,0,5$ ❺. First one must find a basis for the null space of $\mathbf{A}$ ❷, ❸ such as $\mathbf{u}_1=\begin{pmatrix}-1\\1\\0\\0\\0\end{pmatrix}$, $\mathbf{u}_2=\begin{pmatrix}-1\\0\\1\\0\\0\end{pmatrix}$, $\mathbf{u}_3=\begin{pmatrix}-1\\0\\0\\1\\0\end{pmatrix}$ and $\mathbf{u}_4=\begin{pmatrix}-1\\0\\0\\0\\1\end{pmatrix}$, and the observe that $\begin{pmatrix}1\\1\\1\\1\\1\end{pmatrix}$ is an eigenvector for 5. Then using Gram-Schmidt, one gets $\mathbf{P}=\begin{pmatrix}\frac{-1}{\sqrt{2}}&\frac{-1}{\sqrt{6}}&\frac{-1}{\sqrt{12}}&\frac{-1}{\sqrt{20}}&\frac{1}{\sqrt{5}}\\\frac{1}{\sqrt{2}}&\frac{-1}{\sqrt{6}}&\frac{-1}{\sqrt{12}}&\frac{-1}{\sqrt{20}}&\frac{1}{\sqrt{5}}\\0&\frac{2}{\sqrt{6}}&\frac{-1}{\sqrt{12}}&\frac{-1}{\sqrt{20}}&\frac{1}{\sqrt{5}}\\0&0&\frac{3}{\sqrt{12}}&\frac{-1}{\sqrt{20}}&\frac{1}{\sqrt{5}}\\0&0&0&\frac{4}{\sqrt{20}}&\frac{1}{\sqrt{5}}\end{pmatrix}$, which satisfies: $\mathbf{P}^{-1}\mathbf{A}\mathbf{P}=\mathbf{D}$, the diagonal matrix with $0,0,0,0,5$ on the main diagonal ❻.

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Goals:

1) Students will formulate real world situations in meaningful mathematical forms, including frequency interpretation of probability, diagrams or equations.
2) Students will recognize axioms of probability theory and be able to apply them to real-world situations.
3) Students will be able to use counting techniques to compute elementary probability estimates.
4) Students will be able to evaluate probabilities involving continuous density functions and recognize their uses.
5) Students will execute statistical manipulation and computation with random variables and statistical functions in order to solve a posed problem.
6) Students will interpret statistical results about real-world situations.

Assessment Method: Embedded questions throughout homework, quizzes and exams.

Illustration #1: Baye's theorem is very good when applied to diagnosing medical conditions. When the disease being checked for is rather rare, the number of incorrect diagnoses can be surprisingly high.

Consider "screening" for cervical cancer. Let A be the event of a woman having the disease and B, the event of a positive biopsy. So, B occurs when the diagnostic procedure indicates that she does have cervical cancer. Assume that the probability of a woman having the disease, P(A), is 0.0001, that is 1 in 10,000. Also, assume that given that the person has the disease, the probability of a positive biopsy is 0.90. So, the test correctly identifies 90% of all women who do have the disease. Also, assume that if the person does not have the disease, the test incorrectly says that person does have the disease 1 out of every 1000 patients.
1) Find the probability that a woman has the disease given that the biopsy says she does.

The student has to recognize this problem as a Baye's Theorem problem. In order to solve such a problem, the student has to be organized and write down everything that was given in the problem and then write down in symbols what is being asked to solve.

Using A and B as described in the problem, the student should be able to see that the question can be written as: find P(A|B). Also, from the problem, the student can see that
$P(A)=0.0001$, (and therefore $P(A^c)=1-P(A)=1-0.0001=0.9999$).
Also, the student should see that $P(B|A)=0.90$.
Finally, the student should see that $P(B|A^c)=0.001$.

Then, using Baye's Theorem, the student should realize the following is equal to P(A|B):

$$P(A|B) = \frac{P(B|A)P(A)}{P(B|A)P(A) + P(B|A^c)P(A^c)}$$

$$= \frac{(0.9)(0.0001)}{(0.9)(0.0001) + (0.001)(0.9999)}$$

= 0.08

So, only 8% of the women identified as having the disease actually do have the disease. This is an alarmingly low accuracy rate.

2) Now, assume we have a more common disease. Assume that the probability of a woman having the disease, P(A), is 0.01, that is 1 in 100. Now, find the probability that a woman has the disease given that the biopsy says she does. That is, find P(A|B).

$$P(A|B) = \frac{P(B|A)P(A)}{P(B|A)P(A) + P(B|A^c)P(A^c)}$$

$$= \frac{(0.9)(0.01)}{(0.9)(0.01) + (0.001)(0.99)}$$

= 0.90

So, 90% of the women identified as having the disease actually do have the disease. This is a much higher accuracy rate.

These questions combined show the prevalence of a disease has a major influence on the accuracy of the tests.

Illustration #2: Any person who has an IQ in the upper 2% of the general population is eligible to join the international society devoted to intellectual pursuits called Mensa.

1) Assuming IQ's are normally distributed with mean of 100 (i.e., μ=100) and a standard deviation of 16 (i.e., σ=16) what is the lowest IQ that will qualify a person for membership?

Let the random variable Y be the person's IQ. Let y_L be the lowest IQ that qualifies someone to be a member of Mensa. So, the student should see that we are looking for $P(Y \geq y_L)=0.02$.

First, the student should realize that standard normal charts are based on cumulative probabilities. So, the student needs to look up
$P(Y < y_L) = 1 - 0.02 = 0.98$.

Next, the student needs to know how to standardizing the values to read off a standard normal chart.

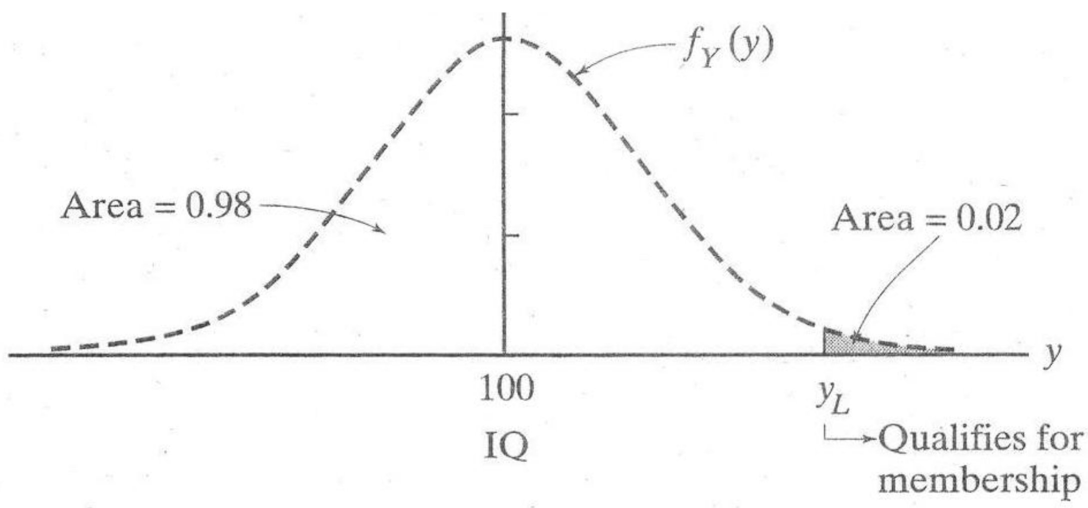
In order to standardize, the student has to realize that $Z = \frac{Y - \mu}{\sigma}$.

Using this, the student should get

$P(Y < y_L)=$ $P(\frac{Y-100}{16} < \frac{Y_L - 100}{16}) = P(Z < \frac{Y_L - 100}{16}) = 0.98$

So, from a standard normal table, the student can see by looking inside the chart, the Z value that corresponds to 0.98 is Z=2.05.

So, setting equal $2.05 = \frac{Y_L - 100}{16}$, we get $y_L=100+16(2.05) = 133$



So, a person needs to get an IQ score of at least 133 to be able to join Mensa.

Illustration #3: As the lawyer for a client accused of murder, you are looking for ways to establish “reasonable doubt” in the minds of jurors. The prosecutor has testimony from a forensics expert who claims that a blood sample taken from the scene matches the DNA of your client. One out of 1000 times such tests are in error.

Assume your client is actually guilty. If six other laboratories in the country are capable of doing this kind of DNA analysis and you hire them all, what are the chances that at least one will make a mistake and conclude that your client is innocent?

Each of the six laboratories constitutes an independent trial, where the probability of making a mistake, p, is 0.001 (i.e., 1 in 1000). Let X=the number of labs that make a mistake (and call your client innocent).

The student should be able to recognize this as a discrete distribution, where X is a binomial with the sample size, n, equal to 6 and p=0.001. So,

P(X=k) = $\binom{n}{k}(0.0001)^k(0.999)^{n-k}$

P(at least 1 lab says the client is innocent)=P(X ≥ 1)

The student should know that the sum of all the probabilities in a discrete distribution always equal 1. So, in this case,
P(X=0)+P(X=1)+P(X=2)+P(X=3)+P(X=4)+P(X=5)+P(X=6)=1

Consequently,
P(X ≥ 1) = P(X=1)+P(X=2)+P(X=3)+P(X=4)+P(X=5)+P(X=6)

Or, equivalently and less computationally intensive,
P(X ≥ 1) = 1 – P(X < 1)
= 1- P(X=0)
$= 1-\binom{6}{0}(0.0001)^0(0.999)^6$
= 0.006.

So, there is only a 0.6% chance of the lawyer’s strategy working.

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Goals:

1 Students will be familiar with the physical significance of the basic partial differential equations of physics: the Laplace equation, the wave equation, and the heat equation.
❷ Students will be able to classify second-order partial differential equations.
❸ They will be able to solve transport equations and understand their connection to wave propagation.
❹ They will understand the significance of the auxiliary conditions of the basic equations: boundary conditions, initial conditions, and mixed conditions.
❺ They will be able to solve these partial differential equations with appropriate boundary conditions on simple domains by the method of separation of variables.
❻ They will become familiar with basic facts of Fourier series, sine series, and cosine series, and be able to utilize them in solving initial and boundary value problems.
❼ Students will understand and be able to contrast the fundamental properties of these basic partial differential equations.

Assessment Method: Embedded questions throughout homework, quizzes and exams.

Illustration #1: Carefully derive the equation of a string in a medium in which the resistance is proportional to the velocity. ❶

Illustration #2: Interpret the physical significance of the homogeneous Neumann boundary condition for the one-dimensional heat equation. ❶❹

Illustration #3: Classify each of the equations. (A list of several second-order equations is given.) ❷

Illustration #4: Solve the first-order equation $2u_t + 3u_x = 0$ with the auxiliary condition $u = \sin x$ when $t = 0$. ❸

Illustration #5: Solve the boundary value problem for the Laplace equation on the rectangle with non-homogeneous boundary condition. ❺❻

Illustration #6: Show that there is no maximum principle for the wave equation. ❼

Illustration #7: Solve the initial value problem for the one-dimensional wave equation for general initial data. ❹❺

General Education and Service Courses

In this appendix, in a similar fashion to the previous, we append the descriptors of the general education and service courses the department offers.

The new pre-baccalaureate courses Math 7, Basic Intermediate Algebra and Math 11, Enhance Intermediate Algebra are used in the prerequisite listings.

The main function of a course is described as either General Education (GE), or Service (S) to a specific major or majors. The on listed first is considered more important.

The courses are:

	Main Function	Clientele	Prerequisites
Math 103	GE	COTA, CLA	ELM or Math 7
Math 112	GE & S	CLA, CHHS, CNSM	ELM or Math 11
Math 114	S & GE	CBA	ELM or Math 7
Math 115	S & GE	CBA	ELM or Math 11
Math 119A	S & GE	CNMS	ELM or Math 11
Math 180	GE	CLA, CHHS	ELM or Math 7
MTED 110	S &GE	CED	ELM or Math 7
NEW GE COURSES These courses have been applied to be GE classes for fall 2007			
Math 101	GE & S	CLA, COTA, CE, CNSM	ELM or Math 7
Math 109	GE	CLA, CHHS	ELM or Math 7
Math 113	S & GE	CNSM, COE, CBA	ELM or Math 11

The initials stand for the colleges: COTA, College of the Arts, CBA, College of Business Administration, CLA, College of Liberal Arts, CED, College of education, COE, College of Engineering, CHHS, College of Health & Human Services, CNSM, College of Natural Sciences & Mathematics

In addition Math 122 and 123 are extensively used as service courses to engineers, some economists and scientists. Those descriptors are in the previous appendix.

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Goals:

❶ Students will formulate real world situations in meaningful mathematical forms, including graphs, tables, diagrams or equations.
❷ Students will execute mathematical manipulation and computation in order to solve a posed problem.
❸ Students will recognize, have knowledge of, be able to combine and evaluate fundamental mathematical expressions and functions such as polynomials and exponentials.
❹ Students will exhibit familiarity and ease of use of basic geometric and arithmetical facts such as the Pythagorean Theorem, similar triangles and ratios.
❺ Students will interpret the mathematical result about real world situations derived mathematically.
❻ Students will be able to use counting techniques to compute elementary probability estimates.

Assessment Method: Embedded questions throughout homework, quizzes and exams.

Illustration #1: Ornithologists have determined that some species of birds tend to avoid flights over large bodies of water during daylight hours, because air generally rises over land and falls over water in the daytime, so flying over water requires more energy.

A bird is released on an island 5 miles from shore. The nesting area is 12 miles down the straight shore from the point on the shore directly opposite the island. The bird uses 10 kcal/mi to fly over land, while it uses 14 kcal/mi to fly over water. Consider the following questions:

5
12

① How much energy will the bird use if it flies directly from the island to the nesting area?

What is expected is that the student will visualize the information in the form of a triangle ❶:

Thus, by letting d denote the distance from the island to the nesting area, the student would arrive, by the use of the Pythagorean Theorem ❹, to

$$d^2 = 5^2 + 12^2 = 25 + 144 = 169,$$

and so one would conclude that $d = 13$ miles. ❷

Naturally, the student should continue to answering the question:

The bird will need $13 \times 14 =$ *182 kcal to accomplish that trip.*❺

② If the bird flies directly over the water into land and then flies over land to the nesting ground, how much energy will the bird need then?

Continuing with the triangle model, the answer is readily arrived at ❶, ❸, ❺:

$$5\times 14+12\times 10=70+120=190 \text{ kcals.}$$

Now the more sophisticated question:

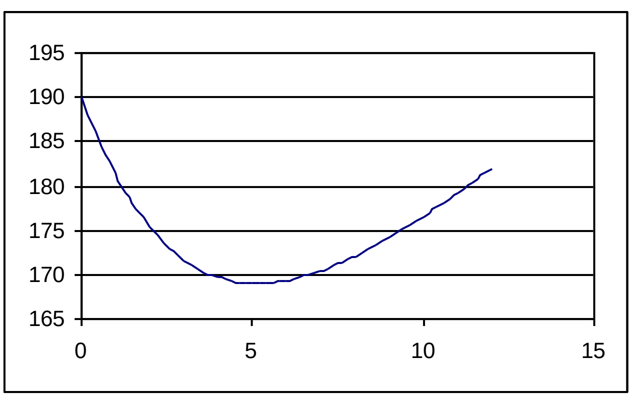
❸ Suppose the bird is to fly directly over the water to some point on the shore between the nearest point and the nesting place, and then fly over land to the nesting place. Can the bird save energy by doing this? If so can the bird just use 170 kcals? If so where should the bird fly?

x

The student is then expected to come up with a variable, x, ❶, ❷ which could represent the distance between the point on shore nearest the island and the point on the shore that the bird will fly to, so the picture now looks like

Now the student is expected to use the Pythagorean Theorem once again, and express the distance y that the bird is flying over water as a function of x:
$y^2=5^2+x^2$, so $y=\sqrt{25+x^2}$ ❸, ❹. Additionally, the distance the bird is flying over land, $z=12-x$. Thus the energy E that the bird will consume when it flies toward the point at x is given by the function:

$$E(x)=14\sqrt{25+x^2}+10(12-x)$$ ❶, ❸.



The student could get a graphical representation of this function: and see a ready answer for the first part of the question ❶, ❸, ❺:

Indeed, the bird can spend less energy ❺ if it flies to some point on shore different from the nearest point and then along the shore.

Also from the picture, the student can identify two destinations that the bird could fly to in order to use exactly 170 kcals. What is needed now is to solve the equation

$$E(x) = 14\sqrt{25+x^2} + 10(12-x) = 170$$ ❷, ❸.

Simplifying, it becomes

$14\sqrt{25+x^2} = 170-120+10x = 50+10x$; or $7\sqrt{25+x^2} = 25+5x$.❷

Squaring both sides,

$49(25+x^2) = (25+5x)^2 = 625+250x+25x^2$.❷

Simplifying one more time,

$24x^2 - 250x + 600 = 0$.❷

Using the quadratic formula,

$x = \frac{250 \pm \sqrt{4900}}{48} = \frac{250 \pm 70}{48} = \frac{320}{48} \text{ or } \frac{180}{48} = \frac{20}{3} \text{ or } \frac{15}{4}$ ❷,

and finally the student should conclude that that the bird should toward a point that is either $6\frac{2}{3}$ miles or $3\frac{3}{4}$ miles away ❺ along the shore from the nearest point to the island.

Ultimately, you would like the student to understand that if the bird flew to any point between the two points obtained in the previous problem, the bird would be using less than 170 kcals.

Those students with an inquiring mind, might develop the curiosity of what is the optimal solution for the bird and how few kcals will it need to arrive at the sanctuary.

Illustration #2: A poker hand, consisting of 5 cards, is dealt from a standard deck of 52 cards. Find the probability that the five cards are in the same suit.

The first issue is that the student will realize that the computation of the denominator of the fraction that will give the probability is of utmost importance, ❶, and that that denominator is $C(52,5) = 2,598,960$ ❻ the number of ways of choosing 5 objects out of 52 objects. Second, the student will compute the numerator of that fraction as $4 \times C(13,5) = 4 \times 1287 = 5,148$ ❻, the 4 coming from the number of choices for the suit, and $C(13,5)$ as the number of ways of choosing 5 cards from the 13 cards in that suit. Taking the ratio, the student will give $\frac{5148}{2598960} = .001980$, or approximately 0.2% as the probability that the five cards are in the same suit ❺.

Alternatively, the student could think of a different model and suggest computing the probability that the five cards are hearts, and then multiplying the answer by 4 ❶. Thus the first card being a heart has probability $\frac{13}{52}$ ❻, and then the probability that the second card is a heart, given that the first one was, is $\frac{12}{51}$ ❻, and then the third one, $\frac{11}{50}$ ❻, and the fourth card, $\frac{10}{49}$, and the final card being a heart: $\frac{9}{48}$. The answer should then be:

$$4\times\frac{13}{52}\times\frac{12}{51}\times\frac{11}{50}\times\frac{10}{49}\times\frac{9}{48}=0.001980$$ ❻, ❺.

Goals:

❶ Students will understand matrices as collectors of information, execute key operations involving matrices, and use these operations to solve relevant problems.
❷ Students will translate real world situations into systems of linear equations, by identifying the unknowns and establishing relations among them. Students will use Gaussian Elimination to find and describe the meaningful solutions to the problem.
❸ Students will be able to use counting techniques to compute probability estimates.
❹ Students will use more sophisticated methods in probability theory such as Random Variables, Bayes' Theorem, Expectation and the Binomial Distribution to solve deeper probability problems.
❺ Students will be familiar with the Central Limit Theorem and how to use it in order to do estimation of probabilities.
❻ Students will, at all times, be able to interpret the mathematical result about real world situations derived mathematically.

Assessment Method: Embedded questions throughout homework, quizzes and exams.

Illustration #1: David is an independent supplier of restaurant supplies. He needs to supply three different customers with vegetables and with meat. The first customer, Mr. Smith requires 20 crates of vegetables and 50 pounds of meat, the second, Ms. Jones, requires 18 crates of vegetables and 60 pounds of meat, while the third one, Mr. Colt requires 30 crates of vegetable and 30 pounds of meat. David can buy the supplies at 4 different wholesalers, Pavilions (P), Vons (V), Albertson's (A) and Ralph's (R). Pavilions charges $12 for a crate of vegetables and $45 for pound of meat. Similarly, Vons charges $15 and $40 respectively for vegetables and meat, Albertson's prices are $18 and $37 for vegetables and meat while those of Ralph's are $26 and $34 respectively. For each of the three orders, David is going to order both vegetables and meat from only one of the four suppliers (in order to get free delivery). How should David order his supplies?

The student should first store the required purchases in a 2×3 matrix where the rows stand for vegetables and meat respectively and the columns for Mr. Smith, Ms. Jones and Mr. Colt respectively: $\begin{pmatrix} 20 & 18 & 30 \\ 50 & 60 & 30 \end{pmatrix}$ ❶. Similarly, then information on the prices should be collected in a 4×2 where the rows stand for Pavilions, Vons, Albertson's and Ralph's and the columns are vegetables and meat respectively: $\begin{pmatrix} 12 & 45 \\ 15 & 40 \\ 18 & 37 \\ 26 & 34 \end{pmatrix}$ ❶. Finally, using matrix multiplication $\begin{pmatrix} 12 & 45 \\ 15 & 40 \\ 18 & 37 \\ 26 & 34 \end{pmatrix}\begin{pmatrix} 20 & 18 & 30 \\ 50 & 60 & 30 \end{pmatrix}$ to obtain the matrix $\begin{pmatrix} 2490 & 2916 & 1710 \\ 2300 & 2670 & 1650 \\ 2210 & 2544 & 1650 \\ 2220 & 2508 & 1800 \end{pmatrix}$ ❶, the student should conclude that for Mr. Smith the groceries should come from Albertson's, for Ms. Jones from Ralph's and for Mr. Colt from either Von's or Albertson's ❶, ❻.

Illustration #2: One rooster is worth five dollars; one hen is worth three dollars; while three young chicks are worth one dollar. Buying 100 fowls with 100 dollars, how many roosters, hens and chicks?

If we let R stand for the number of roosters, H for the number of hens and C for the number of chickens, then the conditions of the problem easily translate to the following two equations ❷:

$$R + H + C = 100$$
$$5R + 3H + \tfrac{1}{3}C = 100$$

R	H	C
12	4	84
8	11	81
4	18	78
0	25	75

Or in augmented matrix notation this becomes $\begin{pmatrix} 1 & 1 & 1 & 100 \\ 5 & 3 & \frac{1}{3} & 100 \end{pmatrix}$ ❶. This matrix reduces ❷ to $\begin{pmatrix} 1 & 0 & \frac{-4}{3} & -100 \\ 0 & 1 & \frac{7}{3} & 200 \end{pmatrix}$ so we have that $R = \frac{4}{3}C - 100$ and $H = 200 - \frac{7}{3}C$. In order for the solutions to make sense, we need to have *C* to be a multiple of 3, and also since $R \geq 0$ and $H \geq 0$, we must have $C \geq 75$ and $C \leq 85$, so the only possibilities for *C* are 75, 78, 81 and 84. In fact, the solutions are ❷, ❻

Illustration #3: Five people: Mr. A, Ms. B, Mrs. C, Mr. D and Mr. E will stand side by side to pose for a picture. What is the probability that Mr. D and Mr. E will not be standing next to each other?

The student should immediately realize that the number of ways that the five people can pose for a picture is $5! = 120$ ❸. The student should also realize that it is easier to count the number of ways Mr. D and Mr. E can stand together, and hence use the complement of what is wanted ❹. So one is reduced to counting the number of ways of arranging A, B, C, D and E for a picture so that D and E stand together. First decision is making D and E stand together, one has two options for it: DE and ED. After that one has to arrange 4 elements: A, B, C, and DE, for which there are 24 ways of doing it, so in total one has $2\times 24=48$ ❸ ways. So there are $120-48=72$ ❸ ways to arrange A, B, C, D and E for a picture so that D and E do not stand together, and hence the probability is $\frac{72}{120}=\frac{3}{5}=60\%$ ❹, ❻.

Illustration #4: A university claims that 85% of its students graduate. One is to test their veracity by setting up a test of their claim. One selects 12 students at random, and sees how many of them graduate. The decision is to accept the school's claim if at least 8 of the 12 students graduated. What is the probability that one comes to the wrong conclusion if indeed the university's claim is true?

Let *Y* denote the number of students among the 12 that graduated, so *Y* is a binomial random variable with $n=12$ and $p=.85$ ❹. Thus one will be wrong if one encounters $Y\le 7$, thus one needs to compute $\mathbf{P}\left(Y\le 7\right)=1-\mathbf{P}\left(Y\ge 8\right)$, and the relevant values are:

8	9	10	11	12
0.068284	0.171976	0.292358	0.301218	0.142242

with a sum of 0.976078, and so one will be wrong 2.39% of the time, a truly negligible possibility ❹, ❻.

Illustration #4: A company manufactures perfume sprayers. They consider that 5% of their production is defective. A random sample of 600 sprayers is tested, which is large enough to treat as a normal approximation to the binomial ❺, ❹. In that sample one expects 30 defective ones ❹.

One also has that the standard deviation $\sigma=\sqrt{600\cdot .05\cdot .95}\approx 5.34$ ❺ atomizers. What is the probability then of each of the following?

① At least 35 defectives? Since one is $\frac{5}{\sigma}=.936$ above the mean, the probability is 17.62% by inspection of the normal table ❺, ❻.

② Between 25 and 35 defectives? Easily, 64.76% ❺, ❻.

③ Suppose one has obtained 50 defectives in the sample—should one worry that perhaps the defectives amount to more than 5%? Since the probability of being 3.7463 standard deviations above the mean is only .02%, one can consider the production to be more defective than was claimed ❺, ❻.

Goals:

- ❶ Students will think statistically as evidenced by problem-solving in statistics.
- ❷ Students will use statistics to create quantitative models of real-world situations, and then use the models to answer questions, make predictions, or make decisions about those real-world situations.
- ❸ Students will demonstrate knowledge of how data is properly collected or generated, including common sources of bias in surveys and experiments.
- ❹ Students will create and interpret numerical summaries and graphical displays of data.
- ❺ Students will decide on appropriate sampling techniques for the purpose of making statistical inferences (i.e., generalize from the sample to the population).
- ❻ Students will communicate the results of a statistical analysis in technical as well as non-technical terms.
- ❼ Students will understand and use confidence intervals and statistical significance, including significance levels and p-values.
- ❽ Students will interpret statistical results in context.

Assessment Method: Embedded questions throughout homework, hands-on activities (Illustration #1, In class hands-on activities), projects (Illustration #4), quizzes and exams (Illustrations #2 & #3).

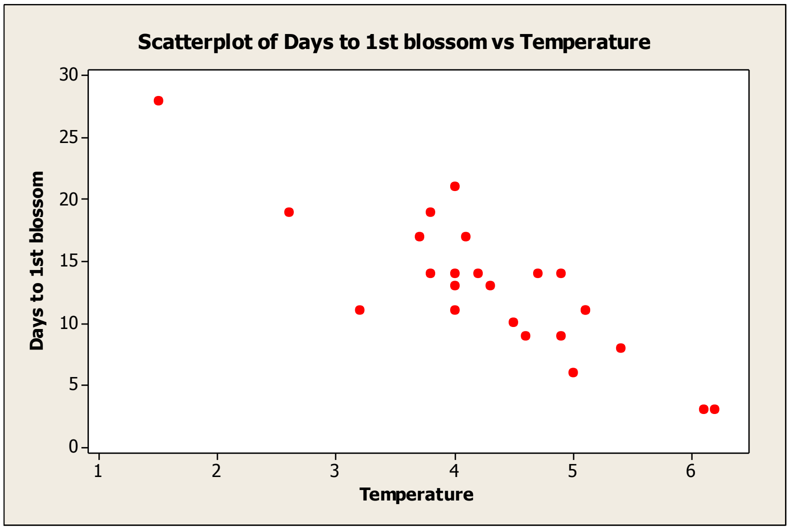
Illustration #1: The anticipation of the first blossoms of spring flowers is one of the joys of April. One of the most beautiful of the spring blossoms is that of the Japanese cherry tree. Experience has taught us that, if the spring has been a warm one, the trees will blossom early, but if the spring has been cool, the blossoms will arrive later. Mr. Yamada is a gardener who has been observing the date in April when the first blossoms appears for the last 25 years. His son, Hiro, went to the library and found the average temperatures for the month of March during those 25 years. The data (Atarashii Sugaku II, 1996) is given in the table below.

Temperature (°C)	4.0	5.4	3.2	2.6	4.2	4.7	4.9	4.0	4.9	3.8	4.0	5.1	4.3
Days in April to 1st Blossom	14	8	11	19	14	14	14	21	9	14	13	11	13

Temperature (°C)	1.5	3.7	3.8	4.5	4.1	6.1	6.2	5.1	5.0	4.6	4.0	3.5	
Days in April to 1st Blossom	28	17	19	10	17	3	3	11	6	9	11	?	

① Make a scatter plot.

What is expected is that the student will enter the data in his/her graphing calculator (TI 83 is recommended for the class) and make a scatter plot. The student should be able to determine that temperature is the independent variable and Days is the dependent variable. ❶, ❹



② Find the equation of the linear regression model if there appears to be a linear relationship between Temperature and Days in April to 1st Blossom.

From the scatter plot the student makes in ①, he/she clearly sees the linear pattern between Temperature and Days in April to 1st Blossom. Using the Calculator, the student finds the equation of linear regression,

$$Days = -4.7 * Temp + 33.1$$ ❶, ❷

③ Find the predicted Days in April to 1st Blossom for the last year's data with the Temperature 3.5 $^{\circ}C$. Interpret the answer.

After getting the equation in ②, the student plugs 3.5 into Temp to get the predicted value for Days in April to 1st Blossom, which is 16.65. The student should be able to round up to 17 Days in April to 1st Blossom to make sense out of it. The student should interpret the answer as "If the average temperature of March of that year is 3.5 $^{\circ}C$, you expect to see the first Cherry Blossom on the 17th of April". ❷, ❻, ❽

④ If the average temperature of March was 0 $^{\circ}C$ due to the unexpected cold, what is the predicted Days in April to the 1st Blossom? Interpret the answer.

By plugging 0 into temp, the student gets 33.1 days. Interpretation would be "If the average temperature of March of a certain year is 0 $^{\circ}C$, then you expect to see the first Cherry Blossom on May 3rd". But the student learned that prediction outside of data range is not recommended. Therefore, the student should write " but, with an average temperature of 0 $^{\circ}C$ in the month of March, it's very likely no Cherry Blossoms will appear that year since it was too cold" as a part of answer as well. ❷, ❻, ❽

Illustration #2: A police-issue radar gun was used to record speeds for randomly selected motorists driving through a 30 mph speed zone. The results from 10 motorists are given below.

31, 30, 33, 43, 26, 37, 30, 28, 45, 36.

① Compute the mean and standard deviation.

Using a calculator, the student computes the mean (33.9 mph) and standard deviation(6.30 mph). ❹

② Construct the 95% confidence interval for μ, the average speed of motorists driving that speed zone.

Using the formula $\bar{x} \pm t_{\alpha/2}(df = n-1)\frac{s}{\sqrt{n}}$, the student compute the confidence interval with significance level α = 0.05. Using the data, the student gets $33.9 \pm t_{0.025}(df = 9)\frac{6.30}{\sqrt{10}}$, (29.39 mph, 38.451 mph). ❼

③ Do they follow the speed limit?

Since the interval includes 30 mph in it, the answer would be "Yes, based on the significance level of 0.05, we conclude that they follow the speed limit". ❶, ❻, ❽

Illustration #3: AT&T reports that 40% of the telephone customers in the city of Long Beach chose AT&T to provide long distance service. A consumer group wants to show that the actual figure is less than that reported by AT&T. A random sample of 300 individual selected from the city of Long Beach yielded 95 chose AT&T. Perform the hypothesis testing using significance level of 0.05.

① Write down the null and alternative hypotheses with a proper definition of parameter.

This is the hypothesis testing on Population proportion. Using standard notations, the student writes

Ho: $p = 0.4$ versus Ha: $p < 0.4$

where the parameter p is the true rate of telephone customers in the city of Long Beach choose AT&T to provide long distance service.

② Compute the test statistic.

The student calculates the test statistic, $z = \frac{\hat{p} - p}{\sqrt{p(1-p)/n}}$ = $z = \frac{95/300 - 0.4}{\sqrt{0.4(1-0.4)/300}}$ = - 2.94. ❹

③ Draw the Decision Rule (Rejection Region).

The student recognizes that this is one-sided hypothesis testing. With α = 0.05, the student has decision rule as reject Ho in favor of Ha if the test statistic in ② is less than -1.645 and do not reject Ho, otherwise. ❼

④ Conclusion with interpretation.

The student writes “With α = 0.05, we reject the null hypothesis (Ho). Therefore, there is enough evidence to say that less than 40% of telephone customers in the city of Long Beach choose AT&T to provide long distance service”. ❼, ❽

⑤ Calculate the *p*-value and perform the *p*-value test. Is the result consistent with what you get in d)?

The student calculates the *p*-value=P(z < -2.94) = 0.0016. Using the *p*-value test, the student rejects the null hypothesis (Ho) since the *p*-value < α and the results from d) and e) are consistent. ❼

Illustration #4: Which Supermarket is the true “Low Price Leader”? (project)
Choose two comparable supermarkets where the student shops mostly, such as Ralph’s and Albertsons, and do a statistical analysis to determine which market is cheaper in general. The following is the procedure. First list 50 items the student buys most often; a half gallon milk, a loaf of bread,…. Using any reasonable sampling scheme, sample 15 items from them. Visit those two supermarkets and check the prices of those 15 items the student sampled. Conduct the analysis using the proper method. There are several things to be mentioned in the project; did you use sale price or regular price for items that were on sale? and did you use the same brand name or equivalent?. The project requires an introduction, raw data sheets, a graphical statistical analysis, a numerical statistical analysis, and a conclusion.
The following is an actual project turned in.

The 50 items the student select are:

1. 15 oz. Cheerios
2. Gallon 1% milk
3. Dozen large eggs
4. 1 lb Farmer John bacon
5. 12 oz. Minute Maid orange juice
6. Loaf Roman Meal bread
7. Package of 8 hot dog buns
8. ½ gallon Breyer’s Ice cream
9. 12 oz. Starkist tuna
10. can of spam
11. 1 lb Oscar Meyer hot dogs.
12. 1 lb Hilshire Farm sausage
13. 20 oz. Kellogg’s Raisin Bran
14. 1 ½ lb Grape Nuts
15. 13 ¼ oz. Lay’s potato chips.

26. 10 ¾ oz. Campbell’s chicken noodle soup
27. 15 oz. VanCamp’s Pork and Beans
28. 6 oz. Contadina tomato paste
29. 24 oz. store brand maple syrup
30. 16 oz. Miracle Whip
31. 1 lb. 12 oz. Del Monte ketchup
32. 1 lb. Blue Bonnet stick margarine
33. 3 oz. Friskies Fancy Feast
34. 1 lb. Medium Red Delicious apples
35. 1 lb. Bananas
36. 5 lbs. Potatoes
37. Head of iceberg lettuce
38. 1 lb. pack of baby carrots
39. 1 cucumber
40. 1 lb yellow onions

16. 2 liter Coca-Cola
17. 2 liter Dr. Pepper
18. 2 liter Mug root beer
19. Lawry's taco seasoning mix
20. 8 oz. sour cream (store brand)
21. 11oz. can of Green Giant corn
22. 1 lb. box of Krispy saltine crackers
23. 1 lb. Wheat Thins crackers
24. 48 Lipton tea bags
25. 10 ¾ oz. Campbell's tomato soup
41. Betty Crocker cake mix
42. 12 oz. Tollhouse chocolate chips
43. 7.25 oz. Kraft macaroni and cheese
44. 2 lb. Uncle Ben's converted rice
45. 2 lb. store brand grape jelly
46. 16 oz. Carnation Coffee mate
47. 16 oz. Crisco all-vegetable shortening
48. 48 oz. Wesson canola oil
49. 1 lb. store brand miniature marshmallows
50. 1 lb. store brand graham crackers

Using the random number table in the back of the textbook, ❸ the student decided to use the last two digits of the first column and went down until the student got 15 different numbers from 01 to 50 (She uses Simple Random Sampling without Replacement) and the numbers were 23, 34, 13, 09, 31, 44, 50, 27, 48, 25, 26, 11, 07, 35, 17. The student then used the grocery items which corresponds to these numbers and found their prices (not sale prices) at Ralph's and at Albertsons. The results of the full prices of 15 items are on the table below:

Grocery Items	Stores and	Respective	Prices:
	Albertsons($)	Ralph's($)	Difference(R-A)($)
7. package of 8 store brand hot dog buns	1.39	1.09	-0.30
9. 12 oz. Starkist tuna	1.99	2.09	0.10
11. 1 lb. Oscar Mayer hot dogs	3.19	3.32	0.13
13. 20 oz. Kellogg's Raisin Bran	3.39	3.39	0.00
17. 2 liter Dr. Pepper	0.99	1.29	0.30
23. 1 lb. Wheat Thins crackers	3.59	3.67	0.08
25. 10 ¾ oz. Campbell's tomato soup	0.68	0.69	0.01
26. 10 ¾ oz. Campbell's chicken noodle soup	0.69	1.25	0.56
27. 15 oz. Van Camp's pork and beans	0.59	0.66	0.07
31. 1 lb 12 oz. Del Monte ketchup	1.31	1.89	0.58
34. 1 lb. Red Delicious apples	0.69	0.99	0.30
35. 1 lb. Bananas	0.59	0.63	0.04
44. 2 lb. Uncle Ben's converted rice	2.51	2.67	0.16
48. 48 oz. Wesson canola oil	2.82	2.79	-0.03
50. 1 lb. store brand graham crackers	2.49	1.99	-0.50
TOTALS	26.91	28.41	1.50

The student used a Bar Graph for graphical presentation and carried the hypotheses testing on Inference in Matched Pairs data and came up with the conclusion that there was no difference in prices between those two markets. ❺

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Goals: Students will:

❶ Exhibit familiarity and ease of use of basic geometric and arithmetical facts such as the Pythagorean Theorem, similar triangles and ratios.

❷ Demonstrate knowledge and understanding of the trigonometric functions, their interrelations and their relations to triangles and the unit circle.

❸ Use the trigonometric functions and their inverses, their manipulation and computation, to solve real world problems involving angles and triangles.

❹ Model a variety of periodic phenomena using the trigonometric functions and their graphs.
❺ Interpret the mathematical result about real world situations derived mathematically.
❻ Exhibit understanding of the arithmetic and geometry of the complex numbers.

Assessment Method: Embedded questions throughout homework, quizzes and exams.

24°
$500\,ft$

Illustration #1: From a point on the ground 500 ft from the base of a building, it is observed that the angle of elevation to the top of the building is 24° and the angle of elevation to the top of a flagpole atop the building is 27°. Find the height of the building and the length of the flagpole.

What is expected is that the student will visualize the information in the form of two triangles ❶:

Then the student would promptly recognize that $\tan 24^{\circ} = \frac{h}{500}$ where h denotes the height of the building ❷, ❸, so straightforward computation gives

$$h = 500 \times \tan 24^{\circ} \approx 500 \times .4452 \approx 223 \text{ feet.}$$

Similarly, if k denotes the height of the building and the flagpole together, we have that ❷, ❸, $\tan 27^{\circ} = \frac{k}{500}$, so

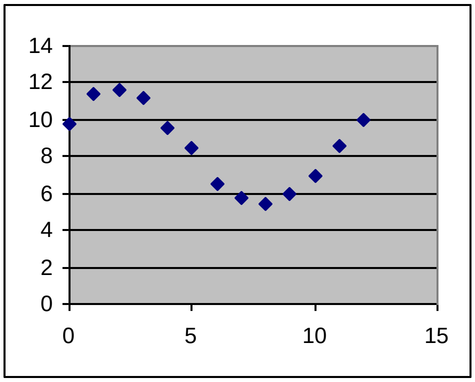
$$k = 500 \times \tan 27^{\circ} \approx 500 \times .5095 \approx 255 \text{ feet.}$$

and finally the height of the flagpole is $255 - 223 = 32$ feet.❺

Illustration #2: The water depth in a narrow channel varies with the tides. Here is the data for one half day:

Time	12	1	2	3	4	5	6	7	8	9	10	11	12
Depth (ft)	9.8	11.4	11.6	11.2	9.6	8.5	6.5	5.7	5.4	6.0	7.0	8.6	10.0

Then the student is expected to answer questions like the following:



① Make a scatter plot of the water depth data where the $x-$ axis is time.

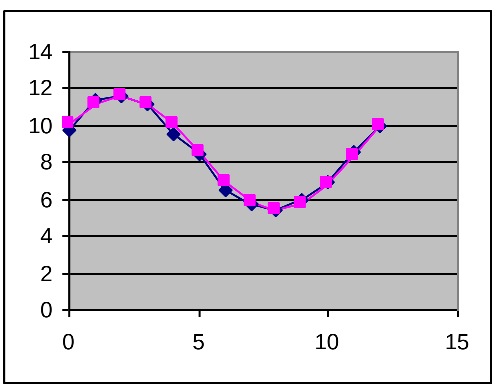
And the student should be thinking ahead that a periodic phenomenon (and hence a trigonometric function) is likely to be involved. ❶, ❹

So the second question should be

② Find a function that models the water depth as a function of time.

The thinking student would select some form of the cosine function (the sine function would do as well) to model it, so if we let $d(t)$ denote the depth of the water measured in feet as a function of time (measured in hours), the student should select an expression of the form:

$$d(t) = a\cos(\omega(t-c)) + b$$



where there are four parameters to be computed, a, b, c and w. The easiest one is b, known as the vertical shift. It represents the average between highest value and lowest value of the curve, so $b = \frac{11.6+5.4}{2} = 8.5$ ❸, ❹. The next is the amplitude, a, which is half of the difference between the highest and lowest value: $a = \frac{11.6-5.4}{2} = 3.1$ ❸, ❹.

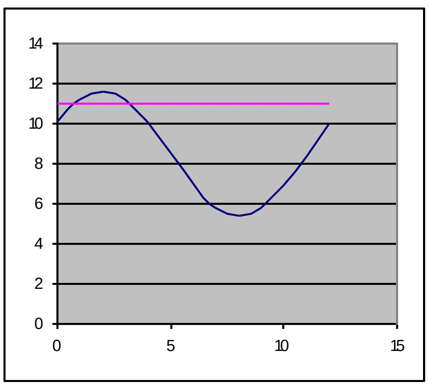
The period is 12 hours since the difference between the highest tide and lowest tide is 6 hours, so $\frac{2\pi}{\omega} = 12$, and so ω satisfies $\omega = \frac{2\pi}{12} \approx .52$ ❸, ❹. Finally, the phase shift, c, is simply the time of the highest depth, so $c = 2$, and we can model using:

$$d(t) = 3.1\cos(.52(t-2)) + 8.5$$ ❸, ❹

and the student observes graphically that the match is acceptable.

Finally, a third and most interesting question would be:

③ If a boat needs at least 11 ft of water to cross the channel, during which times can it safely do so?



Now the student needs to solve $d(t) = 3.1\cos(.52(t-2)) + 8.5 = 11$, which leads to $\cos(.52(t-2)) = .8065$, and so $t = \frac{\arccos(.8065)}{.52} + 2$, $t \approx 3.2$. Since that is the solution in the first quadrant, the other solution is in the fourth quadrant and is given by $t = \frac{-\arccos(.8065)}{.52} + 2$, so $t \approx .8$. Transforming these measurements from hours to minutes, one get that the boat should safely travel between 12:48 and 3:12. ❺

Goals: Students will:

❶ Formulate real world situations in meaningful mathematical forms, including graphs, tables, diagrams or equations, and in words.

❷ Execute mathematical manipulation and computation in order to solve a posed problem.

❸ Recognize, have knowledge of, be able to combine and evaluate fundamental mathematical expressions and functions such as polynomials and exponentials.

❹ Exhibit familiarity and ease of use of basic geometric and arithmetical facts such as the Pythagorean Theorem, similar triangles and ratios.

❺ Interpret the mathematical result about real world situations derived mathematically.

Assessment Method: Embedded questions throughout homework, quizzes and exams.

Illustration #1: The Food and Drug Administration labels suntan products with a sun protection factor (SPF) typically between 2 and 45. Multiplying the SPF by the number of unprotected minutes you can stay in the sun without burning, you are supposed to get the increased number of safe sun minutes. For example, if you can stay unprotected in the sun for 30 minutes without burning, and you apply a product with SPF of 10, then supposedly you can sun safely for 30x10=300 minutes, or 5 hours.

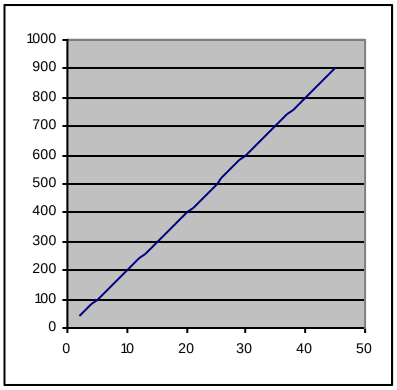
Assume that you can stay unprotected in the sun for 20 minutes without burning.

① Give an equation that gives the maximum safe sun time T as a function of the sun protection factor S.

The student is expected to answer ❶: $T = 20 \times S$.

② Graph your equation. What is the suggested domain for S?

Using either a calculator or simply a hand-sketch, the student should provide the graph ❶:



And also the student explain that the domain consists of the real numbers between 2 and 45, or equivalently, the close interval $2, 45$ ❸.

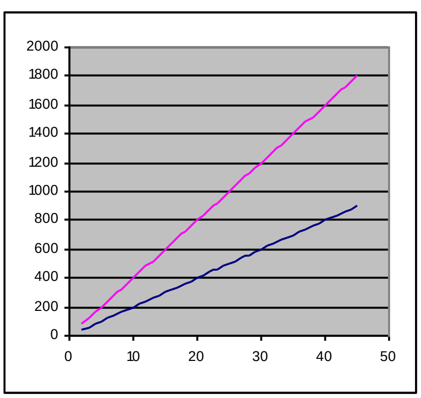
③ Write an inequality that suggests times that would be unsafe to stay out in the sun.

The students should identify that areas above the graph are unsafe, and so their answer should be $T > 20S$ as unsafe ❶, ❺.

④ Suppose one is using the best product around, how many hours can one safely stay out?

The student should realize that 900 minutes is the highest value that the function achieves ❺, and answer “No more than 15 hours”.

⑤ How would the graph change if you could stay unprotected for 40 minutes?



Now the relation between the time and the safety level should given by $T = 40 \times S$, and so the accompanying graph is given in contrast as in the picture.

Perhaps the better student will realize that the slope of the line is of some importance.

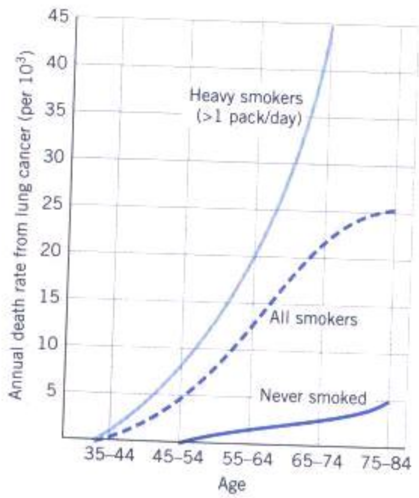


Illustration #2: According to Rubin and Farber's Pathology, "death from cancer of the lung, more than 85% of which is attributed to cigarette smoking, is today the single most common cancer death in both men and women in the United States."

The accompanying graph shows the annual death rate (per thousands) from lung cancer for smokers and nonsmokers.

① The death rate for nonsmokers is roughly a linear function of age. After replacing each range of ages with a reasonable middle age (for example, you could use 60 to approximate the range 55-64), estimate the coordinates of two points on the graph of nonsmokers and construct a linear model. Interpret your results.

Although the question is purposely open, the easiest points to arrive at are possibly $50,0$ and $80,5$, so the linear relationship between $D=$death rate per thousand, and $A=$age in years, is given by $D=mA+b$ where $50m+b=0$ and $80m+b=5$ ❶. Solving then for m and b ❷, ❸ one gets:

$$D=\frac{1}{6}A-\frac{25}{3}.$$

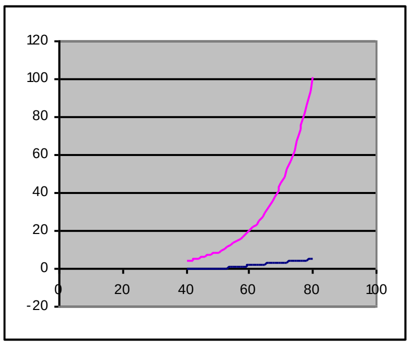
One interesting consequence is then to observe that since the slope of the line is $\frac{1}{6}$, one could conclude that every 6 added years of age leads to one more death in 1000 people ❺.

② By contrast, those who smoke more than one pack per day show an exponential rise in annual death rate from lung cancer. Estimate the coordinates for those points on the graph for heavy smokers and use the points to construct an exponential model (assume a continuous growth rate). Interpret your results.

Now the student should pursue an expression of the form $D = ce^{mA}$ and the two points that one can use to estimate the parameters $60, 20$ and $70, 45$. This leads to the equations: $45 = ce^{70m}$ and $20 = ce^{60m}$ ❷. To solve the equations one needs to look at their quotient ❸: $2.25 = e^{10m}$, and solve for m ❸:

$$m = \frac{\ln 2.25}{10} \approx 0.08109$$

and then substituting in either expression one gets $c \approx 0.1541$. And one finally obtains $D = .1541e^{.08109A}$. One obvious consequence is that by age 80, the death rate becomes the significant: $D \approx 101$ in every thousand people!



③ Graph the models acquired in ① and ②, and compare with the shapes in the original graph.

The graphs are given by

And of course the observation should be made that there is fair resemblance between the original data and the models.

113

Goals: Students will:

- ❶ Formulate real world situations in meaningful mathematical forms, including graphs, tables, diagrams, equations and in words.
- ❷ Execute mathematical manipulation and computation in order to solve a posed problem.
- ❸ Recognize, have knowledge of, be able to combine and evaluate fundamental mathematical expressions and functions such as polynomials and exponentials.
- ❹ Exhibit familiarity and ease of use of basic geometric and arithmetical facts such as the Pythagorean Theorem, similar triangles, and ratios.
- ❺ Interpret and articulate the mathematical result about real world situations derived mathematically.
- ❻ Exhibit proficiency in understanding and usage of sequence and series terminology and language.

Assessment Method: Embedded questions throughout homework, quizzes and exams.

Illustration: Ornithologists have determined that some species of birds tend to avoid flights over large bodies of water during daylight hours, because air generally rises over land and falls over water in the daytime, so flying over water requires more energy.

A bird is released on an island 5 miles from shore. The nesting area is 12 miles down the straight shore from the point on the shore directly opposite the island. The bird uses 10 kcal/mi to fly over land, while it uses 14 kcal/mi to fly over water. Consider the following questions:

5
12

① How much energy will the bird use if it flies directly from the island to the nesting area?

What is expected is that the student will visualize the information in the form of a triangle ❶:

Thus, by letting d denote the distance from the island to the nesting area, the student would arrive, by the use of the Pythagorean Theorem ❹, to

$$d^2 = 5^2 + 12^2 = 25 + 144 = 169,$$

and so one would conclude that $d = 13$ miles. ❷

Naturally, the student should continue to answering the question:

The bird will need $13 \times 14 =$ *182 kcal to accomplish that trip.* ❺

② If the bird flies directly over the water into land and then flies over land to the nesting ground, how much energy will the bird need then?

Continuing with the triangle model, the answer is readily arrived at ❶, ❸, ❺:

$$5\times 14+12\times 10=70+120=190 \text{ kcals.}$$

Now the more sophisticated question:

③ Suppose the bird is to fly directly over the water to some point on the shore between the nearest point and the nesting place, and then fly over land to the nesting place. Can the bird save energy by doing this? If so can the bird just use 170 kcals? If so where should the bird fly?

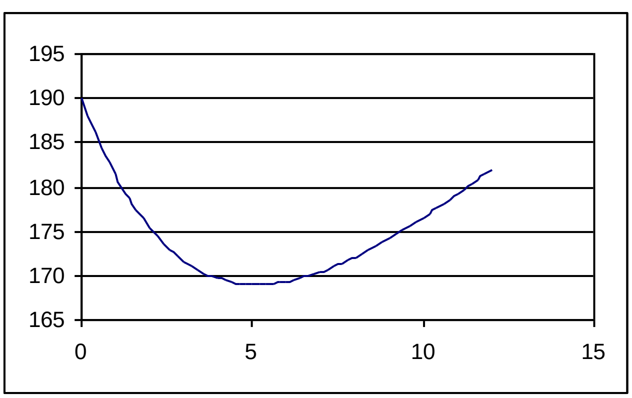
x

The student is then expected to come up with a variable, x, ❶, ❷ which could represent the distance between the point on shore nearest the island and the point on the shore that the bird will fly to, so the picture now looks like

Now the student is expected to use the Pythagorean Theorem once again, and express the distance y that the bird is flying over water as a function of x:

$y^2=5^2+x^2$, so $y=\sqrt{25+x^2}$ ❸, ❹. Additionally, the distance the bird is flying over land, $z=12-x$. Thus the energy E that the bird will consume when it flies toward the point at x is given by the function:

$$E(x)=14\sqrt{25+x^2}+10(12-x)$$ ❶, ❸.



The student could get a graphical representation of this function: and see a ready answer for the first part of the question ❶, ❸, ❺:

Indeed, the bird can spend less energy ❺ if it flies to some point on shore different from the nearest point and then along the shore.

Also from the picture, the student can identify two destinations that the bird could fly to in order to use exactly 170 kcals. What is needed now is to solve the equation

$$E(x) = 14\sqrt{25+x^2} + 10(12-x) = 170$$ ❷, ❸.

Simplifying, it becomes

$14\sqrt{25+x^2} = 170-120+10x = 50+10x$; or $7\sqrt{25+x^2} = 25+5x$.❷

Squaring both sides,

$49(25+x^2) = (25+5x)^2 = 625+250x+25x^2$.❷

Simplifying one more time,

$24x^2 - 250x + 600 = 0$.❷

Using the quadratic formula,

$$x = \frac{250 \pm \sqrt{4900}}{48} = \frac{250 \pm 70}{48} = \frac{320}{48} \text{ or } \frac{180}{48} = \frac{20}{3} \text{ or } \frac{15}{4}$$ ❷,

and finally the student should conclude that that the bird should toward a point that is either $6\frac{2}{3}$ miles or $3\frac{3}{4}$ miles away ❺ along the shore from the nearest point to the island.

Ultimately, the student should understand that if the bird flew to any point between the two points obtained in the previous problem, the bird would be using less than 170 kcals.

Those students with an inquiring mind, might be curious about the optimal flight plan for the bird and how few kcals will it need to arrive at the sanctuary.