Spring 2021 – Algebra Comprehensive Exam Name: ____________

Choose six problems total, including at least two from Part I and two from Part II. Enter the numbers of the problems you want graded here:

Problems							Total
Scores							

Part I: Groups (Choose at least two.)

1. Let G be a finite group.
   (a) Prove that if G is cyclic, then every subgroup of G is cyclic.
   (b) Let H be a normal subgroup of G. Prove that if H is cyclic, then every subgroup of H is normal in G.
   (c) Give an example to show that (b) is false if H is not cyclic.

2. (a) How many conjugacy classes does D_{12}, the dihedral group of order 12, have?
   (b) Show that if G is a finite nonabelian simple group, then for any $x \in G$ with $x \neq e$, the size of the conjugacy class of x is at least 3.
   (c) Let G be a finite group with a normal subgroup H. For $x \in H$, we define the centralizer $C_G(x) = \{g \in G \mid gx = xg\}$ of x in G and the conjugacy class $\mathcal{O}_H(x) = \left\{ hxh^{-1} \mid h \in H \right\}$ of x in H. Prove that $|\mathcal{O}_H(x)| = [HC_G(x) : C_G(x)]$.

3. (a) Prove that there are no simple groups of order 132.
   (b) Give an example of a group of order 132 without a normal Sylow 2-subgroup.
   (c) Give an example of a group of order 132 without a normal Sylow 3-subgroup.

4. (a) Let H be a subgroup of S_n with H not contained in A_n. Prove that $[H : H \cap A_n] = 2$.
   (b) Let G be a finite simple group with a subgroup H such that $[G : H] = k > 2$. Prove that $|G|$ divides $\frac{k!}{2}$.

5. Let G be a finite group. Suppose G has a subgroup $H \neq \{1_G\}$ such that $H \subseteq K$ for every nontrivial subgroup K of G. Label each of the following statements as true or false. Justify each answer with a proof or a counterexample.
   (a) H is a normal subgroup of G.
   (b) H is cyclic of prime order.
   (c) G must be abelian.

Part II: Rings and Linear Algebra (Choose at least two.)

6. Let R be a commutative ring with identity. An element $s \in R$ is called <u>power-stable</u> if $s^2 = s$. An power-stable element s is called <u>trivial</u> if $s = 0$ or $s = 1$; otherwise it is called <u>nontrivial</u>. An element $r \in R$ is called <u>power-vanishing</u> if $r^n = 0$ for some positive integer n.

   (a) Prove that if s is power-stable, then so is $1 - s$.

   (b) Prove that a nontrivial power-stable element cannot be a unit.

   (c) Prove that if R has a unique maximal ideal, then R has no nontrivial power-stable elements.

   (d) Identify the units, zero divisors, power-stable elements, and power-vanishing elements in the ring $\mathbb{Z}/12\mathbb{Z}$.

7. Let R be a commutative ring with the property that for every $r \in R$, there exists $n > 1$ such that $r^n = r$. Prove that every prime ideal in R is maximal.

8. (a) Let R be a commutative ring with identity.

      i. Prove that if the only ideals of R are $\{0\}$ and R, then R is a field.

      ii. Prove that if R has exactly three ideals, then R is not an integral domain.

   (b) Show that $\mathbb{M}_2(\mathbb{R})$, the ring of 2×2 matrices with entries from the real numbers, has no nontrivial proper two-sided ideals but is not a division ring.

9. Let R and S be commutative rings with identity, $\phi : R \to S$ a <u>surjective</u> ring homomorphism. Prove that

   (a) $\phi(1_R) = 1_S$.

   (b) If M is a maximal ideal of R, then $\phi(M)$ is either all of S or a maximal ideal of S.

   Show by example that each of the above may fail if ϕ is a nontrivial homomorphism that is not surjective.

10. Construct a 3×3 matrix having an eigenvalue 1 with corresponding eigenvector $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, an eigenvalue of -1 with corresponding eigenvector $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, and an eigenvalue of 2 with corresponding eigenvector $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$. Is this matrix unique, or could there be others with this property? Justify your answer.